GXBi4

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2000 - HONOURS

Erasmus Students

MA406 - Differential Geometry

Time allowed: *Three* hours.
Full marks for *three* questions.

1. Define (a) a differentiable manifold, (b) a regular surface in $\mathbf{R}^3$ and explain why a regular surface in $\mathbf{R}^3$ is a two-dimensional differentiable manifold.

   Show that the cylinder $\{(x, y, z) \in \mathbf{R}^2 : x^2 + y^2 = 1\}$ is a regular surface and find a parametrisation whose coordinate neighbourhoods cover it.

2. Let $F : U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map on an open subset U of $\mathbf{R}^n$. Define, (a) a critical point of F, (b) a critical value of F, (c) a regular value of F.

   Prove that if $f : U \subset \mathbf{R}^3 \to \mathbf{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in $\mathbf{R}^3$.

   Show that the surface
   $$x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 1$$
   is a regular surface.

3. Let S denote the sphere $x^2 + y^2 + (z-1)^2 = 1$. Define stereographic projection from $S \setminus N$, where $N = (0, 0, 2)$ is the north pole, to the xy-plane. Use stereographic projection to show that S is a differentiable manifold with an atlas consisting of two charts.

4. Define the tangent plane to a (two-dimensional) differentiable manifold S at a point $p \in S$.

   Let X and Y be vector fields on S. Show that $XY - YX$ is also a vector field on S.

   Show that the equation of the tangent plane at the point (x_0, y_0, z_0) of a regular surface $f(x, y, z) = 0$, where 0 is a regular value of f, is
   $$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0.$$