GX 897

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NATIONAL UNIVERSITY OF IRELAND,
GALWAY

SEMESTER 1 (WINTER) EXAMINATIONS 2002-2003

B.Sc. (Honours) : Experimental Physics

Paper III: Quantum Mechanics (EP405)

4EL3-4EP441-3
4BS2-EP444-1

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Time allowed : TWO hours.	Answer THREE questions

Q.1 A linear harmonic oscillator has a potential function $V(x)=\frac{1}{2}kx^2$, where $k=m\omega^2$, and the symbols have their usual meanings.

Show that the Hamiltonian for the system can be written in the following form:

$$H=\frac{\hbar\omega}{2}\left[-\frac{\hbar}{m\omega}\frac{d^2}{dx^2}+\frac{m\omega}{\hbar}x^2\right]$$ *[1 mark]*

By attempting to factorise the Hamiltonian, show that raising and lowering ladder operators exist for this system. *[4 marks]*

Determine the lowest and second lowest energy-eigenfunctions. (You are not required to normalise the eigenfunctions). *[2 marks]*

Determine the lowest energy-eigenvalue of the system. *[2 marks]*

Without going into mathematical detail, state why the lowest energy-eigenvalue cannot be zero. *[1 mark]*

Q.2 Explain how an operator may be represented by a matrix. Show that if the eigenfunctions of the operator are used as a basis for the function-space, the matrix is diagonal. *[2 marks]*

An electron in the *xy*-plane, subject to a two-dimensional linear harmonic potential, has energy-eigenfunctions ψ_i.
E_1 is the non-degenerate lowest energy-eigenvalue with eigenfunction ψ_1.
E_2 is the degenerate second lowest energy-eigenvalue with eigenfunctions ψ_2 and ψ_3.
A weak magnetic field B is applied in the z direction. Using the set ψ_i as basis functions the Hamiltonian matrix is:

$$\begin{pmatrix} E_1 & 0 & 0 & 0 & . & . \\ 0 & E_2 & -i\hbar\, eB/2m & 0 & . & . \\ 0 & i\hbar\, eB/2m & E_2 & 0 & . & . \\ 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}$$

Using the matrix elements given, solve the secular equation to determine the three lowest energy levels in the presence of the magnetic field *[4 marks]*

Find the corresponding energy-eigenfunctions. *[3 marks]*

What is the effect of the magnetic field on the previously degenerate energy levels? *[1 mark]*

Q.3 Write down the Cartesian components of the angular momentum operator: l_x, l_y, and l_z and show that the commutator of l_x and l_y is $i\hbar\, l_z$. *[3 marks]*

Let $l_+ = l_x + i\, l_y$ and $l_- = l_x - i\, l_y$

Noting that $[l_i , l_j] = i\hbar\, \varepsilon_{ijk}\, l_k$, where the symbols have their usual meanings, show that $[l_+ , l_z] = -\hbar\, l_+$ *[1 mark]*

Let ψ be an eigenfunction of l_z with eigenvalue $m\hbar$, where m is an integer in the range $\{ -m_{max}, +m_{max} \}$. Show that $l_+\, \psi$ is also an eigefunction of l_z. How are the eigenvalues of ψ and $l_+\, \psi$ related? *[2 marks]*

Show that $l_- l_+ = l^2 - l_z^2 - \hbar\, l_z$ *[2 marks]*

Determine the eigenvalues of l^2 in terms of m_{max}. *[2 marks]*

Q.4 ψ_n^0 and E_n^0 are the non-degenerate energy-eigenfunctions and energy-eigenvalues for a system with Hamiltonian operator H_0, i.e. $H_0\,\psi_n^0 = E_n^0\,\psi_n^0$

The system is perturbed by a small additional energy term H' such that $H = H_0 + H'$

Show that $E_n^{(1)}$, the first order perturbation contribution to the energy-eigenvalues, is given by $E_n^{(1)} = \langle \psi_n^0 | H' | \psi_n^0 \rangle$

[4 marks]

Obtain an expression for the first order perturbation contributions to the eigenfunctions.

[2 marks]

An infinite potential well is defined as follows:

$x < 0 \qquad V(x) = \infty$

$0 \le x \le L \qquad V(x) = 0$

$x > L \qquad V(x) = \infty$

The energy-eigenfunctions for a particle subject to this potential are:

$$\psi_n = \sqrt{\frac{2}{L}}\,Sin\left(\frac{n\pi x}{L}\right)$$

The system is perturbed by a small energy term $V'(x) = C.(L - x)$ over the interval $0 \le x \le L$. Calculate the first order perturbation contribution to the energy-eigenvalues.

[4 marks]

Note: $\quad Sin^2\theta = \frac{1}{2}(1 - Cos\,2\theta), \qquad \int u\,dv = uv - \int v\,du$

Q.5 Answer *all* parts:

(a) Show that a linear combination of degenerate eigenfunctions is also an eigenfunction. Hence show how an orthonormal pair can be generated from any two degenerate eigenfunctions.

[3 marks]

(b) A is an operator in a two-dimensional vector space represented by the matrix:

$$A = \begin{pmatrix} 1 & 2i \\ -2i & 1 \end{pmatrix}$$

Find the eigenvalues of A. Is A Hermitian?
Find the eigenvectors of A. Find the transformation matrix S which diagonalises A.
Show that S is unitary. Verify that $S^{-1} A\, S$ is diagonal.

[3 marks]

(c) The relativistically correct formula relating energy and momentum is $E^2 = c^2p^2 + m_0^2c^4$. Show how Dirac formulated a relativistically correct differential wave equation, which is first order in time. Comment on the mathematical properties required of any parameters that you introduce.

[4 marks]