OLLSCOIL NA hÉIREANN

THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

WINTER EXAMINATIONS 2002

SECOND ENGINEERING EXAMINATION

MATHEMATICS [MA 250]

Dr D. L. Johnson
Prof. T. C. Hurley
Dr A. Christofides
Dr J. J. Ward

Time allowed: **Three** hours.
Full marks for 5 correct solutions.
Answer **not more than three** questions from each section
Please use **separate answer books** for each section

SECTION A

Q1. (a) Show that $\sum_{k=1}^{\infty} \ln\left(\frac{k+1}{k}\right)$ diverges even though $\lim_{k\to\infty} \ln\left(\frac{k+1}{k}\right) = 0$. Comment.

(b) Using the comparison test, or otherwise, determine whether or not the series

$$\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^3-1}}$$

converges.

(c) Using the ratio test, or otherwise, determine whether or not the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

(d) Define the nth Taylor polynomial $P_n(x)$, for a function $f(x)$, about the point $x = 0$. Calculate $P_5(x)$ for the function $\sec x$ about the point $x = 0$.

Q2. (a) For a curve $\mathbf{r}(s)$ parameterised in terms of the arc–length s, write down the Frenet–Serret formulae for the derivative with respect to s of the unit tangent $\hat{\mathbf{T}}$, unit normal $\hat{\mathbf{N}}$, and the unit binormal $\hat{\mathbf{B}}$.
Equivalently, you may write down the skew–symmetric matrix A for which

$$\frac{d}{ds}\begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{pmatrix} = A \cdot \begin{pmatrix} \hat{\mathbf{T}} \\ \hat{\mathbf{N}} \\ \hat{\mathbf{B}} \end{pmatrix}.$$

(b) For the curve paramaterised in terms of the arc–length s:

$$\mathbf{r}(s) = \arctan s\mathbf{i} + \frac{1}{\sqrt{2}}\ln(s^2+1)\mathbf{j} + (s - \arctan s)\mathbf{k}$$

find
(i) the unit tangent vector $\hat{\mathbf{T}}$;
(ii) the principal unit normal vector $\hat{\mathbf{N}}$.
(iii) Verify that the curvature κ equals $\dfrac{\sqrt{2}}{s^2+1}$.
(iv) Calculate the binormal vector $\hat{\mathbf{B}}$ from $\hat{\mathbf{T}} \times \hat{\mathbf{N}}$.
(v) Show that the torsion τ equals the curvature.

Q3. (a) Show that if $w = f(u,v)$ with $u = x+y, v = 2x-2y$, then

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{\partial w}{\partial u}\right)^2 - 4\left(\frac{\partial w}{\partial v}\right)^2.$$

(b) Find the tangent plane at the point (1, 2, 1) to the surface $xy + yz - 4zx = 0$.

(c) Show that the three surfaces

$$14x^2 + 11y^2 + 8z^2 = 66 \text{ and } 5x - y - 3z^2 = 0 \text{ and } xy + yz - 4zx = 0$$

are mutually perpendicular at the point (1, 2, 1).

Q4. (a) Find the maxima and minima and saddle points (if any) of the function $f(x,y) = xye^{-(x^2+y^2)/2}$.

(b) Find the maximum volume of a cone which can be inscribed in a sphere of radius a, and hence or otherwise show that the ratio of the volume of the cone to that of the sphere is $\left(\dfrac{2}{3}\right)^3$.

Q5. (a) Evaluate the integral $\int\int_R y\,dxdy$ where R is the region bounded by $y = x^2$, $y = x^3$.

(b) By changing the variables and evaluating the Jacobian (or otherwise), find $\int\int_R (x^2+y^2)dxdy$ over the region R in the first quadrant bounded by the curves $xy = 2$, $xy = 4$, $x^2 - y^2 = 1$, $x^2 - y^2 = 9$.

(2002) Second Year Engineering Linear Algebra

1. (a) Determine for each of the following sets of vectors if the given set is a subspace of $\mathbf{R}^3$. (Give a proof in each case).

   i. $\{(x_1, x_2, x_3) \mid 2x_1 + x_2 + x_3 = 0\}$.

   ii. $\{(x_1, x_2, x_3) \mid 2x_1 + x_2 + x_3 = 1\}$.

   iii. $\{(x_1, x_2, x_3) \mid 2x_1 + x_2 + x_3 \geq 0\}$.

   (b) Let $\mathbf{u} = [1, 1, 0, -1]$, $\mathbf{v} = [1, 0, -2, 1]$, $\mathbf{w} = [1, 0, 2, 1]$. Show that the subspace $W = \mathrm{sp}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ of $\mathbf{R}^4$ is the solution space of a homogeneous linear equation $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$ and find the real numbers a_1, a_2, a_3, a_4.

   Let $\mathbf{a} = [3, 1, 0, 1]$. Explain why the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for W but $\{\mathbf{a}, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is not a basis for W.

2. (a) Express the matrix A below as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 4 \\ 0 & -1 & 0 \end{bmatrix}$$

   (b) Find a basis for the column space of the matrix B below from among the columns of B and express the remaining columns as linear combinations of this basis.

$$B = \begin{bmatrix} 1 & 3 & 0 & 1 & 1 \\ 2 & 5 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ -1 & 2 & -4 & 1 & 1 \end{bmatrix}$$

3. (a) Determine for each of the following mappings if the given mapping is a linear transformation. (Give a proof in each case).

   i. $T[x_1, x_2] = [2x_1 + x_2, x_1 + x_2, x_1 - x_2]$.

   ii. $T[x_1, x_2] = [2x_1 + x_2, x_1 + x_2, x_1x_2]$.

   iii. $T[x_1, x_2] = [0, 0, x_1 - x_2]$.

   iv. $T[x_1, x_2] = [0, 0, x_1 - x_2 + 1]$

   (b) Write down the natural matrix representation of the linear transformation

$$T[x_1, x_2, x_3] = [x_1 + x_2 + 2x_3, x_2 + x_3, x_1 + 2x_2 + 4x_3].$$

   Find the matrix representation of T relative to the basis

$$B = \{[1, 1, 0], [-1, 0, 1], [1, 1, 1]\}.$$

pto

4. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 3 \\ 4 & 0 & -1 \end{bmatrix}.$$

Find a matrix Q such that the matrix $Q^{-1}AQ$ is diagonal and hence find the general solution of the system of differential equations

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(Note that 3 is an eigenvalue of A).