GX 2075

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER 1 EXAMINATIONS 2002/2003

THIRD UNIVERSITY EXAMINATION

MATHEMATICS — [MA313]

LINEAR ALGEBRA I

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Time allowed: *Two* hours.
Third Arts Mathematical Studies: Full marks for *four* questions.
All other students: Full marks for *three* questions.

1. (a) What is meant by a *linearly independent set* in a vector space? Determine whether or not

$$\{(-2,2,3),(2,1,-3),(2,4,-3)\}$$

is a linearly independent set in $\mathbb{R}^3$. Is this set a basis for $\mathbb{R}^3$? Deternmine whether or not

$$\{x, x^2, 1+x+x^2\}$$

is linearly independent in $\mathbb{R}[x]$.

(b) What is meant by a *subspace* of a vector space? Decide with proof which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

(i) $\{(x,y,z) : x,y,z \in \mathbb{R}, y = 0\}$
(ii) $\{(x,y,z) : x,y,z \in \mathbb{R}, y \geq 0\}$
(ii) $\{(x,y,z) : x,y,z \in \mathbb{R}, xy = 0\}$

(c) Find a basis for the subspace of all (x,y,z,t,u) in $\mathbb{R}^5$ which satisfy

$$\begin{array}{rcrcrcrcrcl} x & - & 2y & + & 3z & - & 2t & + & u & = & 0, \\ 2x & - & 4y & + & z & - & 2t & + & 2u & = & 0. \end{array}$$

p.t.o.

2. (a) Let $f : V \to W$ be a linear transformation of vector spaces. Explain *kernel* of f, $\ker f$, and *image* of f, $Im\, f$. Show that $Im\, f$ is a subspace of W. Show also that $f(0_V) = 0_W$.

(b) Decide with proof which of the following transformations $T : V \to W$ are linear transformations.

(a) $V = \mathbb{R}^3, W = \mathbb{R}^2$, $T((x, y, z)) = (3x + y, 2y - z)$

(b) $V = \mathbb{R}^3, W = \mathbb{R}^2$, $T((x, y, z)) = (3x + y, 2yz)$

(c) $V = \mathbb{R}^3, W = \mathbb{R}^4$, $T((x, y, z)) = (3x + y, 2y - z, y, 2x - y + z)$

For those which are linear, find the matrix of the linear transformation relative to the natural (standard) bases for the spaces.

3. (a) The matrix A below is the matrix of a linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^3$ relative to the natural (standard) bases for $\mathbb{R}^4$ and $\mathbb{R}^3$.

Find bases for $\ker f$ and $\text{Im } f$ and verify that $\dim \ker f + \dim \text{ Im } f = 4$.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & -2 & 1 & -1 \end{pmatrix}$$

(b) You are given that the following sets W and V are bases for $\mathbb{R}^3$:

$$W = \{\mathbf{u_1} = (1, 1, 2)^t, \quad \mathbf{u_2} = (2, 0, 1)^t, \quad \mathbf{u_3} = (1, 1, 0)^t\}$$

$$V = \{\mathbf{v_1} = (1, 1, -1)^t, \quad \mathbf{v_2} = (1, 0, 1)^t, \quad \mathbf{v_3} = (2, 1, 1)^t\}$$

Find the matrix A of the change of basis from W to V.

State *in terms of this* A the matrix B of the change of basis from V to W. (You need not work out this matrix B explicitly.)

4. (a) Explain *eigenvalue* and *eigenvector* of a square matrix A. Suppose there exists an invertible matrix E and a matrix B such that $E^{-1}AE = B$. Prove that B and A have the same eigenvalues.

If B is a diagonal matrix what does this say about the eigenvalues of A? Show that the columns of E, in this case, are eigenvectors of A.

(b) Let

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

Show that the eigenvalues of A are $\lambda = -2, -2, 4$. Find if it is possible to obtain a nonsingular E and a diagonal D such that $D = E^{-1}AE$ and if so find such an E.