GX 2077

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER I EXAMINATIONS 2002-2003

MA 235 – PROBABILITY AND STATISTICS

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Time Allowed: Two hours
Answer the ten questions in PART A (30 marks)
and
two of the questions in PART B (35 marks each).

PART A

[*Multiple choice*. [30 *marks*]] In each of questions **A1** through **A10**. below, **write down one** choice of answer, and ensure that you write **both the number of the answer and the answer itself**. For example, if in **A1**. below you think **(a)** is the answer, you would write in your answer book

$$\text{A1. (a)} \quad \frac{e^{-10}\frac{10^{40}}{40!} - \binom{1000}{40}(0.01)^{40}(0.99)^{960}}{\binom{1000}{40}(0.01)^{40}(0.99)^{960}} \times 100\%.$$

A1. Suppose that 1% of stock option prices do not change in value when the option is exercised. **What** is the percentage error if we use the Poisson distribution in place of the binomial distribution to approximate the probability that exactly 40 of 1000 randomly selected options will not change in value?

(a) $\dfrac{e^{-10}\frac{10^{40}}{40!} - \binom{1000}{40}(0.01)^{40}(0.99)^{960}}{\binom{1000}{40}(0.01)^{40}(0.99)^{960}} \times 100\%$ **(b)** $\dfrac{e^{-40}\frac{40^{10}}{10!} - \binom{1000}{10}(0.01)^{10}(0.99)^{990}}{\binom{1000}{10}(0.01)^{10}(0.99)^{990}} \times 100\%.$

A2. In a list of 12 stock portfolios, 4 are value-weighted, 4 are price-weighted, and 4 are equal-weighted. If 5 portfolios are randomly selected from the list, let $a = P$(2 are value-weighted) and let $b = P$(2 are value-weighted and 2 are price-weighted). **What** are the exact values of a and b? (*Note*: Of course sampling is without replacement.)

(a) $a = \dfrac{\binom{4}{2}\binom{8}{3}}{\binom{12}{5}}$ and $b = \dfrac{\binom{4}{2}\binom{4}{2}\binom{4}{1}}{\binom{12}{5}}$ **(b)** $a = \binom{12}{5}(\frac{4}{12})^2(\frac{8}{12})^3$ and $b = \binom{12}{5}(\frac{4}{12})^2(\frac{4}{12})^2(\frac{1}{12})$

(c) $a = \binom{5}{2}(\frac{4}{12})^2(\frac{8}{12})^3$ and $b = \binom{5}{2}(\frac{4}{12})^2(\frac{4}{12})^2(\frac{1}{12})$ **(d)** none of these.

A3. Suppose that five of your friends will independently purchase one lottery ticket for the next draw of the National Lottery (in which 6 numbers are drawn from $\{1, 2, ..., 42\}$). **What** is the probability that at least one of these five people will match all six numbers drawn?

(a) $\dfrac{5}{\binom{42}{6}}$ **(b)** $1-\left[1-\dfrac{1}{\binom{42}{6}}\right]^5$ **(c)** $\binom{5}{1}[1-\dfrac{1}{\binom{42}{6}}]\dfrac{1}{[\binom{42}{6}]^4}$ **(d)** $\dfrac{1}{[\binom{42}{6}]^5}$.

A4. Suppose that 10 students will be distributed at random into 2 classes in such a way that each class will get 5 students. If there are 2 whiz kids among the 10 students, **what** is the probability that each class gets one?

(a) $\frac{1}{9}$ **(b)** $\frac{2}{9}$ **(c)** $\frac{3}{9}$ **(d)** $\frac{4}{9}$ **(e)** $\frac{5}{9}$ **(f)** $\frac{6}{9}$ **(g)** $\frac{7}{9}$.

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A5. If five cards will be picked at random from a pack of 52 cards, **what** is the probability that we will obtain two pairs (that is, five face values of the form (x,x,y,y,z) where x,y and z are distinct)?

(a) $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$ (b) $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$

(c) $\binom{13}{4}\binom{12}{1}\binom{4}{3}\binom{4}{1}/\binom{52}{5}$ (d) $\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}/\binom{52}{5}$

(e) $\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$ (f) $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}/\binom{52}{5}$.

A6. The following randomized response technique can be used by an interviewer to elicit answers to sensitive questions. Suppose that we want to estimate $P(I)$, the proportion of executives of Irish financial companies who have offshore bank accounts. We construct 100 flash cards, write "I have an offshore bank account" on 60 of the cards, and write "I do not have an offshore bank account" on the remaining 40 cards. Each executive, in a random sample of 100 executives interviewed, chooses a card at random and, without divulging the question on the card to the interviewer, truthfully responds "yes" or "no" to the statement on the card. [*Note*: An executive will say "yes" if he/she does have an offshore account and the card he/she chooses has "I have an offshore bank account" written on it, and he/she will also say "yes" if he/she does not have an offshore account and the card he/she chooses has "I do not have an offshore bank account" written on it.] Suppose that 30 of the 100 executives respond "yes". **Which** of the equations below do we solve to obtain an estimate p of $P(I)$?

(a) $\frac{70}{100} = \frac{40}{100}p + \frac{60}{100}[1-p]$ (b) $\frac{30}{100} = \frac{40}{100}p + \frac{60}{100}[1-p]$ (c) $\frac{30}{100} = \frac{60}{100}p + \frac{40}{100}[1-p]$.

A7. Box A contains two gold coins, Box B contains two silver coins, and Box C contains one gold and one silver coin. A box is chosen at random and a coin selected from it. If the selected coin is Gold, **what** is the probability that the other coin in the selected box is Gold? (i.e. what is the probability that the selection was made from Box A?)

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$ (e) $\frac{3}{4}$ (f) 1.

A8. On a Friday morning the pro shop of a tennis club has 14 identical cans of tennis balls. If they are all sold by Sunday night and we are interested only in how many were sold on each day, **in how many** different ways could the 14 cans have been sold on Friday, Saturday and Sunday?

(a) $\binom{15}{2}$ (b) 14^3 (c) 3^{14} (d) $\binom{16}{14}$ (e) $\frac{14!}{3!2!1!}$ (f) none of these.

A9. Let X_1 and X_2 be independent random variables satisfying $E(X_1) = 1$, $E(X_2) = 1$, $Var(X_1) = 3$, $Var(X_2) = 2$. **What** is $E(X_1^2 + X_1X_2)$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4 (f) 5.

A10. From telephone records, it is estimated that the proportion of telephone calls that last more than four minutes is e^{-2}. **What** is the probability that a call will last more than 2 minutes? (Assume that the exponential density $f_T(t) = \begin{cases} \frac{1}{\theta}e^{-\frac{1}{\theta}t}, & \text{for } t > 0 \\ 0, & \text{elsewhere} \end{cases}$ is appropriate for the duration T of a random call.)

(a) e^{-1} (b) e^{-2} (c) e^{-3} (d) e^{-4} (e) e^{-5} (f) e^{-6}.

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PART B

B1.

(a) Suppose that 30% of publicly listed bonds will achieve price gains of 20% or more at maturity, that 40% will have a price gain of less than 20% and that 30% will incur a loss. Suppose that you are an investor who chooses at random 5 publicly listed bonds.

(i) [5 *marks*] **Write down** the probability that none of your 5 bonds will incur a loss.

(ii) [5 *marks*] **Write down** the probability that 2 will incur a gain of 20% or more, 2 will gain less than 20% and 1 a loss.

(iii) [5 *marks*] **Find** the conditional probability that all five will incur a loss given that at least four will incur a loss.

(b) Let X have the Poisson(θ) density $P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$.

Answer (i) OR (ii) below.

(i) [5+5 = 10 *marks*] **Show** that $E(X) = \lambda$ and $Var(X) = \lambda$.

OR

(ii) [10 *marks*] **Show** that the binomial density $P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}, x = 0, 1, \ldots, n,$ converges to the above Poisson density as n tends to ∞ and θ tends to 0 in such a way that $n\theta := \lambda$ is fixed, .

(c) During its opening hours, the number of bad checks that a bank receives during any time period $(0, t)$ has a Poisson distribution with on average 0.4 bad checks per hour. Thus the density function of X is

$$P(X = x \text{ in time period of length } t) = e^{-0.4t}\frac{(0.4t)^x}{x!}, \quad x = 0, 1, 2, \ldots.$$

(i) [2 *marks*] **Write down** the probability of no bad check in the first two hours of a random business day.

(ii) [3 *marks*] Use the fact that the time T until the first bad check appears has the exponential density

$$f_T(t) = \begin{cases} 0.4e^{-0.4t}, & \text{for } t > 0 \\ 0, & \text{elsewhere} \end{cases}$$

to **re-calculate** the probability of no bad check in the first two hours of a random business day.

(iii) [5 *marks*] **Find** the hazard function $h(t) = \frac{f(t)}{1 - F(t)}, \quad t > 0,$ of T.

B2.

(a) [10 *marks*] **Find** the density function of $Y = X^3$ where X has the $N(\mu, \sigma^2)$ density function

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{-\tfrac{1}{2\sigma^2}(x-\mu)^2\right\}, \quad -\infty < x < \infty.$$

(b) Modern theories concerning the relationship between risk and return, along with empirical evidence, suggest that the average rate of return for stocks listed in the over-the-counter (OTC) market exceed those of the more mature companies in the New York Stock Exchange (NYSE). Suppose that the annual return X_1 (in %) on OTC stocks and the annual return X_2 (in %) on NYSE stocks are independent and that $X_1 \sim N(12, 16)$ and $X_2 \sim N(9, 9)$.

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