GX 2081

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

WINTER EXAMINATIONS 2002

B.A. and B.Sc. Degree Examination
Higher Diploma in Mathematics Examination

MA341 - Metric Spaces

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Time allowed: *two* hours.
Answer *three* questions.

1. (a) Define the terms *metric space* and *contraction mapping.*

   (b) Let $C[0,1]$ be the metric space of continuous functions $x\colon [0,1] \to \mathbf{R}$ with metric $d(x,y) = \max\{|x(t) - y(t)| : t \in [0,1]\}$. Prove that $g(x)(t) = 1 + \int_0^t u^3 x(u)\,du$ is a contraction mapping on $C[0,1]$.

   (c) Use the Contraction Mapping Theorem to deduce that the differential equation

   $$\frac{dx}{dt} = t^3 x \quad (0 \le t \le 1), \quad x(0) = 1$$

   has a unique solution in $C[0,1]$. (You may assume that $C[0,1]$ is complete.)

   (d) Let $x_0(t) = 1$ for all $t \in [0,1]$ and set $x_{i+1} = g(x_i)$. Determine the functions $x_1(t)$, $x_2(t)$, $x_3(t)$ and guess the function x to which the sequence $x_0, x_1, \cdots$ converges.

2. (a) Determine an interval $[-1,b]$ and a constant $0 < K < 1$ such that the formula $g(x) = (x^3 + x^2 - 1)/6$ defines a differentiable function $g\colon [-1,b] \to [-1,b]$ with $|g'(x)| \le K$ for all $x \in [-1,b]$. Then use the Mean Value Theorem to prove that $g(x)$ is a contraction mapping on $[-1,b]$.

   (b) Use the Implicit Function Theorem to prove that the equation

   $$4xt - 3\cos(x) = 1 - 2t\cos(t)$$

   uniquely defines a continuous function on $[1,\infty)$. Can the same argument be applied to the interval $[0,\infty)$?

   (c) What is a *complete* metric space? By means of an example, explain why the space of continuous functions from $[0,1]$ to $\mathbf{R}$ is not complete with respect to the metric $d(x,y) = \int_0^1 |x(t) - y(t)|\,dt$.

3. (a) State and prove the Contraction Mapping Theorem.

   (b) State and prove the Implicit Function Theorem.

   (c) Give an example of a function $f\colon (0,1] \to (0,1]$ which has no fixed point and which is a contraction with respect to the usual metric.

4. (a) Let S^2 denote the unit sphere in $\mathbf{R}^3$ endowed with the "shortest path" metric. Let Δ denote the area of a geodesic triangle on S^2 with interior angles α, β, γ. Prove that

$$\alpha + \beta + \gamma = \pi + \Delta.$$

   (b) Let $\mathbf{H}^2$ denote the set of complex numbers with positive imaginary part. Let $PSL(2, \mathbf{R})$ denote the set of functions

$$T(z) = \frac{az+b}{cz+d}$$

   with $a, b, c, d \in \mathbf{R}$, $ad - bc > 0$. Prove that if $z \in \mathbf{H}^2$ then $T(z) \in \mathbf{H}^2$.

   (c) Find a function $T \in PSL(2, \mathbf{R})$ which maps both $-1+\mathbf{i}$ and $1+\mathbf{i}$ to the imaginary axis.

   (d) Calculate the distance between $-1+\mathbf{i}$ and $1+\mathbf{i}$ with respect to the hyperbolic metric on $\mathbf{H}^2$.

5. Let (X, d) be a metric space.

   (a) Prove that all contraction mappings and all isometries $f : X \to X$ are continuous.

   (b) Show that

$$|d(x,z) - d(y,u)| \leq d(x,y) + d(z,u)$$

   for all $u, x, y, z \in X$. Then show that for any $a \in X$ the function

$$f\colon X \to \mathbf{R}, x \mapsto d(a,x)$$

   is continuous with respect to the usual metric on $\mathbf{R}$.

   (c) Prove that X is complete if it is compact.

   (d) Prove that if $f\colon X \to X$ is continuous and A is a closed subset of X then $f^{-1}(A)$ is also a closed subset of X.