Gx 2082

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER 1 EXAMINATIONS, 2002-2003

THIRD UNIVERSITY EXAMINATION IN ARTS & SCIENCE

MATHEMATICS [MA343] - GROUPS 1

HONOURS

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Time allowed: *Two* hours.
Answer three questions.

1. (i) List the elements of the symmetric group S_3 of all permutations of $\{1, 2, 3\}$ and calculate its multiplication table.

   (ii) Write down the axioms for a multiplicative group.

   (iii) Let G be a finite (multiplicative) group. Show that a non empty subset H of G is a subgroup if it is closed. Deduce that the set A_n of even permutations is a subgroup of the symmetric group S_n.

p.t.o.

2. (i) The finite abelian (commutative) group G has subgroups H and K such that $G = HK$ and $H \cap K = 1$. Show that the map $f : H \times K \to G$ defined by $f(h,k) = hk \quad (h\epsilon H, k\epsilon K)$ is an isomorphism.

(ii) Consider $U_{15} = \{1, 2, 4, 7, 8, 11, 13, 14\}$ which is a group under multiplication mod 15. Determine (the elements of) the cyclic subgroups $\langle 11 \rangle$ and $\langle 2 \rangle$ and show that $U_{15} = \langle 11 \rangle \times \langle 2 \rangle$. Is this group cyclic? Why (not)?

(iii) Show that $C_3 \times C_4$ is cyclic.

3. (i) Let K be a normal subgroup of the finite multiplicative group G. Define *normal* and show that the set G/K of cosets of K is a group under (set) multiplication. Show also that the map $f : G \to G/K$ given by $f(x) = Kx \quad (x\epsilon G)$ is a homomorphism.

(ii) Let $Q = \{\pm 1, \pm a, \pm b, \pm c\}$ be the quaternion group of order 8, with $a^2 = b^2 = c^2 = abc = -1$, and take $K = \{\pm 1\}$. Show that K is a normal subgroup of Q, and find the multiplication table of Q/K.

4. Let a be an element of the finite group G.

(i) Show that the centraliser $C_G(a) = \{x\epsilon G : ax = xa\}$ is a subgroup of G.

(ii) Define $ccl(a)$, the conjugacy class of a in G, and show that

$$|G| = |C_G(a)|.|ccl(a)|$$

(iii) Show that the permutation $(1\,3\,4)$ has 8 conjugates in the symmetric group S_4, and find its centraliser. How many conjugates does $(1\,3\,4)$ have in the alternating group A_4?