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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER I EXAMINATIONS 2002 – 2003

MA 387 – PROBABILITY

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Time Allowed: **Two** hours.

Answer question 1 [40 *marks*] **and any two of questions 2– 4** [30 *marks each*].

1. [40 *marks*] **Insert** *numerical* answers in each of the 11 questions **A.** through **K.** below. Part marks will *not* be allotted in any part, so any work you may exhibit will *not* be examined. *Full marks for* 10 *correct answers*.

   A. There are 3 randomly selected individuals in a room. **What** is the probability that at least two of them have the same birthday (i.e. born on the same month and day). [*Note*: Ignore leap years, so assume 365 days in a year, and assume that each person has a probability of $\frac{1}{365}$ of being born on any particular day.]

   B. Suppose that 10 students will be distributed at random into 2 classes in such a way that each class will get 5 students. If there are 2 whiz kids among the 10 students, **what** is the probability that each class gets one?

   C. Suppose that you have one line of six numbers for the next draw of the National Lottery, in which six numbers will be drawn at random from the set of integers $\{1, 2, \ldots, 42\}$, and then a seventh 'bonus' number is drawn. **What** is the probability that you match 3 of the first 6 numbers drawn and the bonus number?

   D. Refer to Question C. above. Suppose that two of your friends will independently purchase one lottery ticket for the next draw of the National Lottery. **What** is the probability that at least one of these two people will match all of the first 6 numbers drawn?

   E. If five cards will be picked at random from a pack of 52 cards, **what** is the probability that we will obtain two pairs? [*Note*: A 'two pair' hand means five face values of the form (x,x,y,y,z) where x,y and z are distinct.]

   F. If you hold 2 tickets in a lottery for which 5 tickets were sold and 3 prizes are to be given, **what** is the probability that you will win at least one prize?

   G. A fair coin will be flipped until we have obtained 4 heads. **What** is the probability that exactly 10 flips will be required?

   H. *Áine*, *Fáinne* and *Gráinne* will take turns flipping a fair coin, with *Áine* being the first to flip. The winner is the one of the three who first obtains a head. **What** is *Áine's* probability of winning?

   I. Suppose that the number of goals scored in a hurling game has a Poisson distribution with, on average, 2 goals per 30 minutes. **What** is the probability of one goal being scored in 60 minutes?

   J. In a game of Yahtzee, five fair six-sided dice are rolled simultaneously. **What** is the probability of getting four of a kind (i.e. the same number on exactly four sides).

   K. If Tom and Mary are among 10 people who are arranged at random in a line, **what** is the probability that there are exactly 4 people between them?

continued ⇒

2.

(a) [5 *marks*] Answer exactly one of the two parts (i) and (ii) below.

(i) By thinking of a counting problem involving binomial coefficients, prove that $n!!$ is divisible by $((n-1)!!)^n$. *Note*: $m!!$ means $(m!)!$

OR

(ii) **Write down** the binomial expansion of $(1+x)^n$ and use it to **prove** that $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ and that $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots$

(b) [10 *marks*] Prove exactly one of the three results (i), (ii) and (iii) below.

(i) Let X represent the number of "successes" we will obtain when n items are taken without replacement from a population that consists of N items, of which a are of one kind (success) and the remaining $N-a$ are of a second kind (failure). **Show** that for each fixed $x = 0,1,2,\ldots,\min(a,n)$, the hypergeometric probability $P(X=x) = \dfrac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$ converges to the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ as N tends to ∞ and a tends to ∞ in such as way that the proportion of successes in the population $\frac{a}{N}$ tends to θ.

OR

(ii) **Show** that for each fixed $x = 0,1,2,\ldots,n,$ the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ converges to the Poisson probability $e^{-\lambda}\dfrac{\lambda^x}{x!}$ as n tends to ∞ and θ tends to 0 in such as way that $\lambda := n\theta$ remains fixed.

OR

(iii) Let X have the binomial density $P(X=x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0,1,\ldots,n$. **Show** that $E(X) = n\theta$ and $Var(X) = n\theta(1-\theta)$.

(c) Suppose that the three assumptions underlying a Poisson process hold. That is,
($A1$) in a small time interval or amount of space the probability of occurrence of the event of interest is proportional to the size of the interval, i.e. $P(1 \text{ arrival in } [t,t+\Delta t]) = \alpha\Delta t$,
($A2$) the probability of two or more occurrences in a small time or space interval is negligible compared with the probability of one occurrence, and
($A3$) occurrences in non-overlapping intervals are independent.
Let $f(x,t) = P(x \text{ arrivals in the time interval } (0,t))$.

(i) [5 *marks*] **Show** that $f(x,t+\Delta t) = f(x,t)(1-\alpha\Delta t) + f(x-1,t)(\alpha\Delta t)$.
Hint: Start by copying the following into your answer book:

$$f(x,t+\Delta t) = P(x \text{ successes in the time interval } (0,t+\Delta t]) =$$
$$P(x \text{ arrivals in } (0,t) \text{ and } 0 \text{ arrivals in } [t,t+\Delta t]) +$$
$$P(x-1 \text{ arrivals in } (0,t) \text{ and } 1 \text{ arrival in } [t,t+\Delta t]) +$$
$$P(x-2 \text{ arrivals in } (0,t) \text{ and } 2 \text{ arrivals in } [t,t+\Delta t]) + \ldots .$$

Then note that by ($A2$) you can ignore all but the first two terms on the right side of this equation.

(ii) [5 *marks*] Hence **show** that $\dfrac{df(x,t)}{dt} = \alpha[f(x-1,t) - f(x,t)]$. (*)

(iii) [5 *marks*] **Show** that the Poisson density

$$f(x) = P(X=x) = e^{-\alpha t}\frac{(\alpha t)^x}{x!}, \quad x = 0,1,2,\ldots$$

satisfies the infinite system of differential equations (*) above.

continued ⇒

3.

(a) Suppose that a fair six-sided die will be rolled twice. Let A_1 be the event that an even number shows on the first roll, let A_2 be the event that an even number shows on the second roll, and let A_3 be the event that both rolls result in the same number.

(i) [7 *marks*] **Write down** the values of
$P(A_1), P(A_2), P(A_3), P(A_1 \cap A_2), P(A_1 \cap A_3), P(A_2 \cap A_3)$ and $P(A_1 \cap A_2 \cap A_3)$.

(ii) [3 *marks*] Hence **state with explanation** if A_1, A_2 and A_3 are independent. *Hint*: Recall that three events A_1, A_2 and A_3 are independent if they are pairwise independent [i.e. $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all $i,j = 1,2,3,\ i \neq j$] *and* $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$.

(b) In medical diagnostic testing, the probability that a diseased individual will have a positive test result is called the *sensitivity*, or *true positive rate* (*TPR*) of the test. The probability that a disease-free individual will have a positive test result is called the *false positive rate* (*FPR*) of the test. Suppose now that the proportion of Irish people with cancer is $\frac{1}{200}$, and that a new cancer screening test has a *TPR* of 0.95 and an *FPR* of 0.01.

(i) [5 *marks*] **What** proportion of cancer patients who take the test will have a positive result.

(ii) [5 *marks*] Given that a patient has a positive result, **what** is the probability that he/she has cancer?

(c) A fair coin will be tossed repeatedly.

(i) [5 *marks*] **Write down** the probability that in 10 flips we will obtain exactly 4 heads.

(ii) [5 *marks*] Let $X = \#$ flips until we have obtained the first head.
Find the conditional probability $P(X = 3 \mid X \geq 2)$.

4.

(a) Suppose that n men check in their hats at a cloakroom and that their hats are returned at random (so that the probability is $\frac{1}{n}$ that any particular man gets back his own hat).

i. [10 *marks*] **Show** that the probability that none of the men gets back his own hat is
$$p_n := 1 - \left\{1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1}\frac{1}{n!}\right\}.$$

ii. [5 *marks*] Hence **write down** a formula for the probability that exactly r ($\leq n$) of the men get back their own hats. Your answer should involve n, r and p_k where you must give k in terms of n and r.

(b) There are six people in a room, of whom three are lecturers (Pat, Tom and Mary) and the remaining three are students. Three people will be selected at random from the room

i. [2 *marks*] **Write down** the probability that none of the three lecturers will be selected.

ii. [6 *marks*] **Calculate** the probability that Pat, Tom and Mary will all be selected given that at least one of them will be selected.

(c) [7 *marks*] Suppose that 5 distinguishable objects are placed into 5 cells so that all 5^5 possible arrangements are equally likely. **Find** the number of these that will have exactly one cell empty.