GX 2087

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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

WINTER EXAMINATIONS 2002

B.Sc. (Part II) EXAMINATION

MATHEMATICS [MA 484]

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Time allowed: **Two** hours.
Full marks for 3 correct solutions.

Q1. (a) Let the random Variable X have probability density function $f_X(x|\theta)$, depending on the parameter θ. Let $\widehat{\theta}$ be an estimator of θ based on a random sample $X_1, X_2, \ldots, X_n$ of size n.
Define the following terms:

1. Unbiased estimator
2. Consistent estimator
3. Minimum variance unbiased estimator (MVUE)
4. Mean square error of an estimator
5. Asymptotically unbiased estimator

(b) Prove that if $\widehat{\theta}$ is asymptotically unbiased and $\lim_{n\to\infty} Var(\widehat{\theta}) = 0$, then $\widehat{\theta}$ is a consistent estimator of θ.

(c) Let $\widehat{\theta}$ be an unbiased estimator of θ and let U be an unbiased estimator of 0, with $\operatorname{cov}(\widehat{\theta}, U) \neq 0$. Show that there exists an $a \neq 0$ such that $\phi_a := \widehat{\theta} + aU$ has smaller variance than $\widehat{\theta}$ and is also unbiased for θ. Hence, or otherwise **prove** that $\widehat{\theta}$ is MVUE for $\theta \Leftrightarrow \widehat{\theta}$ is uncorrelated with all unbiased estimators of 0.

(d) Let $X \sim I_{(\theta,\theta+1)}$, ie X have uniform distribution on $(\theta, \theta+1)$. Show that $X - \frac{1}{2}$ is unbiased for θ, but is not MVUE, by examining the variance of $X - \frac{1}{2} + \frac{\sin 2\pi X}{2\pi}$.

P. T. O.

Q2. (a) State and prove the Cramér–Rao lower bound for the variance of an unbiased estimator $t(X_1, X_2, \ldots, X_n)$ of a parameter θ, based on a random sample $X_1, X_2, \ldots, X_n$ of size n.

(b) Suppose that t_1 is an unbiased and efficient estimator for θ, and that t_2 is also unbiased, but has efficiency e, satisfying $0 < e < 1$. Show that the correlation coefficient $\rho_{t_1,t_2} = \sqrt{e}$.

(c) Let X_1, X_2 be a random sample of size 2 where X has density

$$f(x|\theta) = \frac{3x^2}{\theta^3} I_{(0,\theta)}(x).$$

Using squared error loss, calculate the risk of $t_1(X_1, X_2) = \frac{2}{3}(X_1 + X_2)$ and $t_2(X_1, X_2) = \frac{7}{6}\max(X_1, X_2)$.
For which value of c does the estimator $t^*(X_1, X_2) := c \cdot \max(X_1, X_2)$ have smallest risk?
How do these risks compare with the value yielded by the formula in part (a)? Comment.

Q3. (a) Discuss the method of maximum likelihood as applied to the problem of estimating the parameter of a distribution $f(x|\theta)$. Are maximum likelihood estimators necessarily unique? What are the "optimality" properties enjoyed by maximum likelihood estimators?

(b) Let the random variable Y have $N(\mu, \sigma^2)$ distribution. The random variable $X := e^Y$ is said to have **lognormal**(μ, σ^2) distribution. Using the invariance property of maximum likelihood estimators, obtain the maximum likelihood estimators of e^{μ} and e^{σ^2} based on a random sample $X_1, X_2, \cdots, X_n$ of size n.

(c) Using the method of moments, show how it is possible to get two different estimators of the two parameters μ, σ^2 of a **lognormal**(μ, σ^2) distribution.
$(M_Y(t) := E(e^{tY}) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.)$
The following data was obtained from a random sample of size 10 from a **lognormal**(μ, σ^2) with both parameters unknown:

19.88	6.04	12.81	42.52	42.73
69.40	12.30	9.68	16.78	5.31

Calculate the two different method of moments estimates for both μ and σ^2.

P. T. O.

Q4. (a) Define the following terms:

1. loss function
2. risk function
3. mean risk
4. minimax decision function
5. Bayes decision function with respect to the prior density $\pi(\theta)$ of θ.

Prove that a Bayes estimator for a parameter θ having constant risk, is also a minimax estimator.

(b) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a Bernoulli(p) population. Let the parameter p have a Beta(a, b) prior density, ie:

$$\pi(p) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}.$$

Using a quadratic error loss function, obtain the Bayes estimator of p. Hence, or otherwise, show that

$$\frac{\sum_{i=1}^{n} X_i + \frac{\sqrt{n}}{2}}{n + \sqrt{n}}$$

is the minimax estimator for p.
Using a sample size of 16 and a quadratic error loss function, for what values of p will the minimax estimator have smaller risk than $\overline{X}$?