GX 2089

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

CHRISTMAS EXAMINATIONS 2002

B.Sc. (Part II) EXAMINATION

MATHEMATICS — [MA423]

FOURIER TRANSFORMS

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Time allowed: **Two** hours.
Answer *three* questions.

1. (a) Compute the appropriate Fourier transforms of each of the following functions:
   (i) $f\colon \mathbb{R} \to \mathbb{C}$, $f(x) = 1$ on $[-\alpha, \alpha]$ and $f(x) = 0$ outside this interval.
   (ii) $f\colon \mathbb{T}_{2\pi} \to \mathbb{C}$, $f(x) = 0$ on $[0, \pi)$ and $f(x) = 1$ on $(\pi, 2\pi)$.
   (iii) $f\colon \mathbb{Z} \to \mathbb{C}$, $f[n] = 2^{-|n|}$.
   (iv) $f\colon \mathbb{P}_4 \to \mathbb{C}$, $f = (2, 0, -2, 1)$.

   (b) Prove the validity of the synthesis formula for the Discrete Fourier Transform for functions on $\mathbb{P}_N$.

2. (a) Let f be an absolutely summable function on $\mathbb{Z}$ (i.e., $\sum_n |f[n]| < \infty$), with Fourier transform $F(s)$. Show that the synthesis formula is valid for f.

   (b) Let f be a continuous function on $\mathbb{T}_p$ (i.e., f is a continuous p-periodic function on $\mathbb{R}$), with Fourier coefficients $F[k]$. Prove Bessel's Inequality:

$$\sum_{k=-\infty}^{\infty} |F[k]|^2 \leq \frac{1}{p} \int_0^p |f(x)|^2 \, dx .$$

p.t.o.

3. Answer **either** part (a) **or** part(b):

(a) Let f be a twice-differentiable function on $\mathbb{T}_p$. Prove that the Fourier series $\sum_{k=-\infty}^{\infty} F[k]e^{2\pi ikx/p}$ converges to $f(x)$ at every point.

(b) The de la Vallèe-Poussin Power Kernels are given by

$$\delta_n(x) = \frac{4^n}{p}\binom{2n}{n}^{-1} \cos^{2n}(\pi x/p) .$$

State the basic properties of these functions and prove that, if f is a continuous p-periodic function, then the sequence of trigonometric polynomials defined by

$$T_n(x) = \int_{-p/2}^{p/2} f(u)\delta_n(x-u)\,du$$

converges uniformly to f.

4. (a) Let g be a continuous p-periodic function on $\mathbb{R}$ with Fourier coefficients $G[k]$. If g is sampled at $N = 2M+1$ points to produce the vector

$$\gamma = \big[g(0), g(p/N), g(2p/N), \ldots, g\big((N-1)p/N\big)\big] ,$$

explain why the Discrete Fourier Transform of this vector is close to the vector

$$[G[0], G[1], \ldots, G[M], G[-M], \ldots, G[-2], G[-1]]$$

of Fourier coefficients of g when N is large.

(b) Give the definition of the convolution product $f * g$ for suitable functions f, g on each of the domains $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$ and $\mathbb{P}_N$. For *one* of these domains show that the convolution product corresponds to pointwise (or coordinatewise) mutliplication of the respective Fourier transforms. Discuss briefly the application of convolution to filtering.

p.t.o.

5. (a) Let $N = 2M$ and let f be a function on $\mathbb{P}_N$. Show that the computation of the Discrete Fourier Transform of f can be reduced to the computation of the M-dimensional transforms of the vectors containing the even and odd components, respectively, of f.

(b) Explain how the above result can be used recursively when N is a power of 2 to yield the Fast Fourier Transform. Illustrate your answer by considering the case $N = 8$.

Fourier Transform Formulas

	Synthesis	Analysis
$f : \mathbb{R} \to \mathbb{C}$	$f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i s x}\, ds$	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx$
$f : \mathbb{T}_p \to \mathbb{C}$	$f(x) = \sum_{k=-\infty}^{\infty} F[k] e^{2\pi i k x/p}$	$F[k] = \frac{1}{p} \int_0^p f(x) e^{-2\pi i k x/p}\, dx$
$f : \mathbb{Z} \to \mathbb{C}$	$f[n] = \int_0^p F(s) e^{2\pi i s n/p}\, ds$	$F(s) = \frac{1}{p} \sum_{n=-\infty}^{\infty} f[n] e^{-2\pi i s n/p}$
$f : \mathbb{P}_N \to \mathbb{C}$	$f[n] = \sum_{k=0}^{N-1} F[k] e^{2\pi i k n/N}$	$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-2\pi i k n/N}$