GX 2092

OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

EASTER EXAMINATIONS 2003

THIRD UNIVERSITY EXAMINATION

MATHEMATICAL METHODS [MM351]

Dr. Dave Johnson,
Professor T. C. Hurley,
Dr. R. A. Ryan
Dr. R. Dark

Time allowed: *Three* hours.
Answer *three* questions from each section.
Use separate answer books for each section.

SECTION A

1. (a) Define the terms *Eulerian* and *Hamiltonian* as applied to graphs. Show that, in an Eulerian graph, every vertex has even degree.

(b) Describe the *Bridges of Königsberg Problem* and show that it has no solution.

(c) Describe the *Knight's Tour Problem* and show that there is no solution on a 4×4 board.

2. (a) What is a *tree*? What is a *spanning tree* in a graph? Use *Prim's Algorithm* to find a minimum spanning tree for the weighted graph defined by the following table:

	A	B	C	D	E	F
A		7	15	11	7	10
B	7		11	18	3	12
C	15	11		27	8	13
D	11	18	27		18	20
E	7	3	8	18		9
F	10	12	13	20	9	

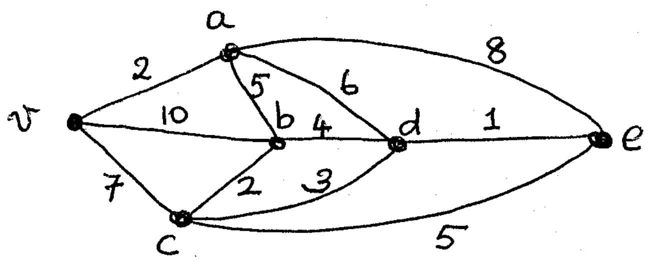
pto

(b) Answer **either** (I) **or** (II):

(I) Using a binary decision tree, explain why any algorithm that sorts a list of n numbers must require $O(n \log_2 n)$ operations. (You may use Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$.)
Describe the *Heapsort* algorithm and use it to sort the following list into ascending order:

$$17, 3, 9, 1, 11, 10, 8, 12, 4.$$

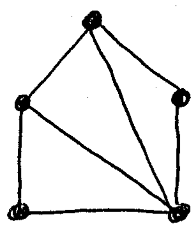
(II) Apply the *Shortest Path Algorithm* to find the shortest path from vertex v to vertex e in the following graph:



3. (a) Prove *Euler's Formula*, $v - e + f = 2$, for a connected planar graph. Kuratowski's Theorem says that every non-planar graph contains one or other of two particular non-planar graphs. Sketch these two graphs.

(b) Answer **either** (I) **or** (II):

(I) Define the *Chromatic Polynomial*, $P_G(k)$, of a graph G, and explain how it can be used to determine the *chromatic number* of G. Find the chromatic polynomial for the graph shown below:



(II) State and prove the *Art Gallery Theorem*. Sketch a gallery for which four guards (and no fewer) are required to watch all the walls.

pto

4. (i) Sketch the images of the angular sector $0 \leq \theta \leq \pi/4$ (where $z = re^{i\theta}$) under the maps:

(a) $w = iz$; (b) $w = z^2$; (c) $w = iz^2$; (d) $w = 1/z$.

(ii) Find the image of the lower half-plane $y \leq 0$ (where $z = x + iy$) under the Möbius transformation

$$w = \frac{z+i}{z-i}.$$

(iii) Show that the image of a straight line or a circle under a Möbius transformation

$$w = \frac{az+b}{cz+d} \quad \text{(where } ad \neq bc\text{)}$$

is a straight line or a circle.
[You may assume that a straight line or circle is given by an equation of the form $pz\overline{z} + az + \overline{a}\,\overline{z} + q = 0$, where p and q are real and a is complex.]

5. (i) Give the definition of a *harmonic* function of 2 variables, and show that if $z = x + iy$ and if the function $f(z) = u(x, y) + iv(x, y)$ is differentiable, then $u(x, y)$ and $v(x, y)$ are both harmonic.
[You may assume the Cauchy-Riemann equations.]
Deduce that the inverse tangent function $v(x, y) = \tan^{-1}(y/x)$ is harmonic.

(ii) Find a Möbius transformation which sends $-1, i, 1$ to $0, i, \infty$ respectively.
Hence find a harmonic function $v(x, y)$ defined on the unit disc $x^2 + y^2 < 1$ such that

$$v(x,y) = \begin{cases} \pi/2 & \text{on the upper semi-circle } x^2+y^2=1,\ y>0, \\ -\pi/2 & \text{on the lower semi-circle } x^2+y^2=1,\ y<0. \end{cases}$$

Section B

MM351

Professor B. Straughan;
Dr. M. S. Ó Confhaola;
Professor M. F. McCarthy;
Dr. P. M. O'Leary.

6. Show that the transformation **T** given by

$$\frac{1}{15}\begin{bmatrix} -5 & -14 & 2 \\ -10 & 5 & 10 \\ 10 & -2 & 11 \end{bmatrix}$$

is an orthogonal transformation. A vector field **A** is defined in the x-frame by

$$A_1 = x_1^2, A_2 = x_2^2, A_3 = x_3^2.$$

Evaluate the field in the x'-frame and verify that div**A** is an invariant.

7. Let ε_{ijk} represent the alternating symbol. Evaluate the following determinant:

$$\begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$$

and hence evaluate $\varepsilon_{ijk}\varepsilon_{rst}$.
Show that
(a) $\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$;
(b) $\varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir}$;
(c) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$;
(d) $\varepsilon_{ijk}\varepsilon_{ijm}a_m = 2a_k$.

8. Let $u(x,t)$ denote the temperature in a rod occupying $0 < x < 1$ at some time $t > 0$. This temperature is governed by the following initial boundary value problem for the heat flow equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1$$

$$u(0,t) = u(1,t) = 0, \quad t > 0$$

$$u(x,0) = 4x(1-x), \quad 0 \leq x \leq 1.$$

Determine the temperature distribution in the rod.

9. Let $u(r,\theta)$ denote the steady state temperature in the quadrant $0 < r \leq a$, $0 \leq \theta \leq \pi/4$, where (r,θ) are the usual plane polar coordinates and a is a constant. The steady state temperature is governed by the following boundary value problem for Laplace's equation:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, \; 0 < r \leq a, \; 0 \leq \theta \leq \pi/4$$

$$u(a,\theta) = \theta, \; 0 \leq \theta \leq \pi/4$$

$$u(r,\theta) = 0, \quad \theta = 0, \quad \theta = \pi/4, \quad 0 < r \leq a$$

Solve this problem using the method of separation of variables.

10. Find the solution of the one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

in the region $0 \leq x \leq L, t > 0$ which satisfies the boundary conditions

$$y(0,t) = y(L,t) = 0, \text{ for all } t > 0$$

and the initial conditions

$$\frac{\partial y(x,0)}{\partial t} = \frac{4V_0}{L^2}(Lx - x^2).$$

$$y(x,0) = 0$$

where V_0 is a constant.
Determine the total energy in the string.

11. Consider a long uniform wire of density ρ which is fixed at $x = 0$ and stretches to infinity under a uniform tension T. While at rest, the wire receives a blow from a mallet which imposes a transverse velocity $x/100$ over $0 \leq x \leq 1$ and zero everywhere else. Using separation of variables and the Fourier Integral determine the transverse displacement $u(x,t)$.