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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2003

FIRST SCIENCE EXAMINATION

MATHEMATICS

MA 103 — *ALGEBRA*

PASS

Second Paper

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Time allowed: *three* hours.
Answer *six* questions only.

1. Let $A = \begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix}$.

(a) Find AB.

(b) Find A^{-1} and B^{-1}.

(c) Show that $B^{-1}A^{-1} = (AB)^{-1}$.

(d) Show that $|AB| = |A||B|$ (where $|A|$ is the determinant of A).

2. For the transformation of the plane defined by the matrix $\begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}$, find

(a) the image of the point $(1, -1)$,

(b) the image of the line $2x + y = 2$,

(c) the line whose image is $x + y = 0$.

p.t.o.

3. Let

$$A = \begin{pmatrix} 6 & -3 \\ 5 & -2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find a diagonal matrix D and an invertible matrix E such that $AE = ED$.

(c) Calculate A^{100}.

4. (a) Reduce the conic $9x^2+18x+4y^2-16y-11=0$ to standard form and sketch its graph.

(b) Prove by induction that

$$\begin{pmatrix} 11 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 11^n & \frac{11^n-1}{10} \\ 0 & 1 \end{pmatrix}.$$

5. (a) Indicate on Argand diagrams the set of points which satisfy

(i) $z+\overline{z}=2$,

(ii) $-\frac{\pi}{4} \leq \operatorname{Arg}(z) \leq \frac{3\pi}{2}$,

(iii) $1 \leq |z-(1+i)| \leq 2$.

(b) Calculate

$$\left(\frac{5+i}{3-2i}\right)^{10},$$

expressing your answer in the form $a+ib$.

(c) Find all complex numbers z such that $z^4+2z=0$.

6. (a) Show that the nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$ where

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

Display these roots in an Argand diagram when $n=5$.

(b) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z+z^{-1} = 2\cos\theta$.

(ii) By expanding $(z+z^{-1})^4$ and using de Moivre's theorem, show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

p.t.o.

7. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 0 & -4 \\ 1 & 2 & 5 \end{pmatrix}.$$

(a) Find the adjoint A^* of A.

(b) Compute the product AA^* and state the determinant of A.

(c) Find A^{-1}.

(d) Use (c) to solve the following.

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & -2 \\ 3x & & & - & 4z & = & 22 \\ x & + & 2y & + & 5z & = & -18 \end{array}$$

8. Every year in a certain country, $\frac{1}{5}$ of the people resident in the capital city move outside the capital, and $\frac{1}{10}$ of the people resident outside the capital move in. Let x_n, y_n denote the fraction of the country's population resident inside and outside the capital, respectively, after n years (thus $x_n + y_n = 1$). Assume the total population of the country remains constant.

(a) Find the transition matrix T for this process.

(b) Show that 1 and 0.7 are eigenvalues of T, and find a corresponding eigenvector for each eigenvalue.

(c) Explain the term "steady state", and find the steady state in this problem.

(d) Show that x_n and y_n tend to the steady state values as $n \to \infty$, regardless of the values of x_0 and y_0.