

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2003

FIRST ARTS EXAMINATION

MA123 - ALGEBRA

Dr D. Johnson

Prof T. Hurley

Ms D. Quin

Dr G. Ellis

Time allowed: *three* hours.

Answer *five* questions.

1. Consider the matrices

$$A = \begin{pmatrix} 5 & -1 \\ 7 & -3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 5 \\ 4 & 7 \end{pmatrix}$$

(a) Calculate:

(i) A^{-1} , the inverse of A , and

(ii) B^{-1} , the inverse of B

(iii) the product AB .

Verify $(AB)^{-1} = B^{-1}A^{-1}$.

(b) Use A^{-1} to solve the system (i) and the matrix equation (ii):

(i) $5x - y = 25$

$7x - 3y = 27$

(ii) $XA = B$

2. Consider the matrix

$$A = \begin{pmatrix} 3 & -7 \\ 1 & -5 \end{pmatrix}.$$

(a) Write down the linear transformation defined by A . Find the image under this transformation of:

(i) the point $(3, 1)$

(ii) the line $x + 2y = 5$.

(ii) Find the eigenvalues and eigenvectors of A . Hence, write down a diagonal matrix D and a non-singular (invertible) matrix E such that $AE = ED$.

Deduce that $A = EDE^{-1}$ and hence find A^4 .

3. (a) Show by induction that 4 divides $9^n - 1$ for all $n \geq 1$.
 (b) Use the Euclidean Algorithm to calculate $\gcd(539, 168)$ and express it in the form $539s + 168t$ with s and t in \mathbb{Z} .
 Find integers x and y such that $539x + 168y = 49$ and show that there are no integers x and y such that $539x + 168y = 50$.

4. A major telephone company (company X) has studied the tendency of users to switch from one carrier to another. The company has established that, over successive twelve-month periods, 20% of their customers will switch to a competing service while 30% of those who use the competing service will switch to X. Let x_n and y_n denote the proportions using company X and the competing service in year n (with x_0 and y_0 the initial proportions).

- (i) Find the transition matrix, A , corresponding to the process such that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{and show that } A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

- (ii) Find the eigenvalues and eigenvectors of A and hence calculate A^n .
 (iii) If company X presently controls 70% of the market, what percentage can it expect to control in the long term?

5. (a) Calculate $8^{-1} \bmod 29$.

- (b) Calculate $A^{-1} \bmod 29$ where $A = \begin{pmatrix} 3 & 7 \\ 1 & 5 \end{pmatrix}$. Express each of the four entries in A^{-1} as an integer from 0 to 28.

- (c) The ciphertext

VHEMEHJSETXE

is written in a 29-letter alphabet ($A=0, \dots, Z=25, _=26, ?=27, !=28$) and was produced by applying the function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 3 & 7 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

to pairs of letters. Determine the *first two* letters of plaintext.

p.t.o.

6. (a) Calculate A^{-1} where $A = \begin{pmatrix} 2 & 4 & 9 \\ 1 & 2 & 4 \\ 2 & 3 & 8 \end{pmatrix}$.

(b) Solve the following system of linear equations.

$$\begin{aligned} 2x + 4y + 9z &= 3 \\ x + 2y + 4z &= 1 \\ 2x + 3y + 8z &= 3 \end{aligned}$$

(c) Use the equality $(AB)^t = B^t A^t$ to prove that $(A^t)^{-1} = (A^{-1})^t$. Then solve the following system of linear equations.

$$\begin{aligned} 2x + y + 2z &= 3 \\ 4x + 2y + 3z &= 5 \\ 9x + 4y + 8z &= 12 \end{aligned}$$

7. (a) Find the modulus and argument of the complex number

$$w = \frac{4(1 + \sqrt{3}i)}{1 - \sqrt{3}i}$$

and then express w^3 in the form $x + iy$ with x, y real numbers.

(b) Factorize $x^5 - 1$ as a product of real linear and quadratic factors. Then deduce that

$$\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}.$$