G02104

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2003

FIRST ENGINEERING & INFORMATION TECHNOLOGY EXAMINATION

MATHEMATICS [MA150]

MA153 — *ALGEBRA*

Second Paper

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Time allowed: *Three* hours.
Answer <u>six</u> questions.

1. (a) Solve, if possible, the following system of linear equations in x, y, z, and w. Give the general solution.

$$\begin{array}{rcrcrcrcr} x & + & 2y & - & z & + & 3w & = & 3 \\ 2x & + & 4y & + & 4z & + & 3w & = & 9 \\ 3x & + & 6y & + & z & + & 8w & = & 10 \end{array}$$

(b) Determine the values of k so that the following system in x, y and z has:
(i) a unique solution,
(ii) no solution,
(iii) infinitely many solutions.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 1 \\ 2x & + & 3y & + & kz & = & 3 \\ x & + & ky & + & 3z & = & 2 \end{array}$$

In cases (i) and (iii) determine the general solution.

p.t.o.

2. Let

$$A = \begin{bmatrix} 1 & -4 & 1 \\ 2 & -3 & 2 \\ 5 & 1 & 0 \end{bmatrix}.$$

(a) Find the adjoint A^* of A.

(b) Calculate the product AA^*.

(c) Using (a) and (b), state the determinant of A and find A^{-1}.

(d) Find the 3×3-matrix B with $BA = A^2 - 2A$.

3. (a) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$. Compute the matrix products AB and BA.

(b) Let

$$X = \begin{bmatrix} 1 & -4 & 1 & -1 \\ 0 & 3 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

Use the Gauss-Jordan method to calculate the inverse of X. Check that your answer is correct by verifying that $X^{-1}X = I_4$.

4. (a) Show that

$$\frac{1}{13}\begin{bmatrix} -3 & 12 & 4 \\ 4 & -3 & 12 \\ 12 & 4 & -3 \end{bmatrix}$$

is an orthogonal matrix.

(b) Show that if $U = [u_{ij}]$ is an upper triangular $n \times n$ matrix, then $\det(U) = u_{11}u_{22}\dots u_{nn}$.

p.t.o.

5. (a) Using de Moivre's Theorem, find integers a, b and c such that

$$\text{Cos}\, 5\,\theta = a\,\text{Cos}^5\,\theta + b\,\text{Cos}^3\,\theta + c\,\text{Cos}\,\theta$$

for all θ.

(a) Find the square roots of $5 + 12i$.

(c) In the Argand plane sketch the set of points z such that $|z+1| + |z-1| = 4$.

(d) Prove that

$$1 + \frac{1}{3}\text{Cos}\,\theta + \frac{1}{9}\text{Cos}\,2\,\theta + \ldots + \frac{1}{3^k}\text{Cos}\,k\,\theta + \ldots.$$

$$= \frac{9 - 3\,\text{Cos}\,\theta}{10 - 6\,\text{Cos}\,\theta}$$

6. (a) Write down the 12^{th} roots of unity in the form $a+ib$, $a, b \in \mathbb{R}$, $i = \sqrt{-i}$ and illustrate them in the Argand plane. Show also that the 12^{th} roots are

$$w_k = \text{Cos}\,\frac{2k\pi}{12} + i\,\text{Sin}\,\frac{2k\pi}{12} \quad \text{for} \quad 0 \le k \le 11.$$

(b) Factorize $z^6 + 1$ (which is equal to $(z^{12}-1)/(z^6-1)$) into a product of three quadratic factors that are irreducible over $\mathbb{R}$.

(c) Divide $z^{12} - 1$ by $z - 1$ and by letting $z = 1$ deduce that

$$(1 - w_1)(1 - w_2)(1 - w_3)\ldots(1 - w_{11}) = 12$$

where

$$w_k = \text{Cos}\frac{2k\pi}{12} + i\,\text{Sin}\frac{2k\pi}{12} \quad 1 \le i \le 11.$$

7. (a) Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & -1 & 2 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Write down a diagonal matrix D and a matrix E such that $E^{-1}AE = D$.

(b) Let B be a matrix with eigenvalue λ. Let X be an invertible matrix. Show that λ is also an eigenvalue of $X^{-1}BX$.

(c) Let B be an invertible matrix with (non-zero) eigenvalue λ. Show that λ^{-1} is an eigenvalue of B^{-1}.

8. (a) Find the vertex, focus, axis and direction x of the parabola

$$x = y^2 + 3y + 3$$

and sketch the parabola.

(b) Prove that the tangents to a parabola at the extremities of a chord through the focus are perpendicular to each other.

(c) Find an orthogonal transformation of the axes which reduces the conic

$$xy = 1$$

to standard form. Sketch the conic and state what type of conic it is.