GX 2105

OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2003

FIRST UNIVERSITY EXAMINATION

MATHEMATICS [MA180]

MA181 - Analysis

HONOURS

First Paper

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Time allowed: *Three* hours.
Full marks for six questions.

Q1. (a) Find all the asymptotes and sketch the graph of the function $\dfrac{1}{x^2 - x}$.

(b) Evaluate the following limits:

(i) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$ (ii) $\displaystyle\lim_{x\to 2} \frac{4-x^2}{3-\sqrt{x^2+5}}$ (iii) $\displaystyle\lim_{x\to 0} \frac{1-\cos 2x}{\sin 2x \sin 3x}$.

Q2. (a) Find the derivative $\dfrac{dy}{dx}$ in each of the following cases:

(i) $y = \tan(\sqrt{\sec x})$ (ii) $y^2(\sin x) + y = \tan^{-1} x$ (iii) $y = x(\sin(\log x) - \cos(\log x))$.

(b) A billboard is to be made with 100m^2 of printed area and with margins of 2m at the top and bottom and 4m on each side. Find the outside dimensions of the billboard if its total area is to be a minimum.

Q3. (a) Evaluate *three* of the following integrals:

(i) $\displaystyle\int \frac{x+1}{\sqrt{x^2+1}}dx$ (ii) $\displaystyle\int \frac{dx}{\sqrt{2x-x^2}}$ (iii) $\displaystyle\int \tan^4 x\, dx$ (iv) $\displaystyle\int \frac{dx}{e^x+1}$

(b) Derive the reduction formula:

$$\int \sin^n x\, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

and use it to evaluate $\int \sin^5 x\, dx$.

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Q4. (a) Let A be a subset of the real line, R. Define the following terms:
A is *finite*; A is *infinite*; A is *countable*; A is *uncountable*.
Give examples of a countable set A and an uncountable set B. *Prove* that B is uncountable.

(b) (i) Define $|x|$ and show that $|x+y| \leq |x| + |y|$.
(ii) Evaluate $\int_0^3 |x^2 - 4|\, dx$.

Q5. (a) Prove that if a sequence is increasing and bounded above then it converges.

(b) The sequence (a_n) is defined by $a_1 = 2$, $a_{n+1} = \sqrt{a_n + 6}$ $(n \geq 1)$.
(i) Calculate a few terms and guess the behaviour of the sequence.
(ii) Given that the sequence converges, find its (positive) limit.
(iii) Now show that part (a) applies, so that the sequence does indeed converge.

Q6. (a) What does it mean to say that the function f is continuous at c in (a, b)? Use the limit theorems to show that f defined by

$$f(x) = x^5 - 3x^4 + 1$$

is continuous at 2.

(b) State the Intermediate Value Theorem and use it to show that the polynomial in part (a) has a root in $[-1, 0]$ and another in $[0, 1]$.

(c) Outline a proof of the Intermediate Value Theorem.

Q7. (a) The function g is defined on $(1, 3)$ by

$$g(x) = \begin{cases} px^2 + x + 3 & \text{if } 1 < x \leq 2 \\ 3x + q & \text{if } 2 < x < 3 \end{cases}$$

Show that g is continuous on $(1, 3)$ if and only if $q = 4p - 1$. What are the values of p and q if g is differentiable on $(1, 3)$?

(b) (i) Prove the Mean Value Theorem, assuming Rolle's Theorem.
(ii) The function f satisfies $f'(x) > 0$ for all x in $(1, 2)$. Show that f is increasing on $(1, 2)$.

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Q8. (a) (i) Show, using the definition

$$\ln x = \int_1^x \frac{1}{t}dt \quad (x > 0),$$

that $\ln ab = \ln a + \ln b \ \ (a, b > 0)$.
Explain briefly why given c there exists $d > 0$ such that $\ln d = c$, and hence define the exponential e^x.

(ii) Calculate the lower Riemann sum $L(h, P)$ where $h(x) = \frac{1}{x}$ and P is the partition of $[1, 3]$ into eight equal parts. What does the answer say about the value of e?

(b) Show that the improper integral

$$\int_1^\infty \frac{dx}{\sqrt{x+1}}$$

diverges.

Show also that

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} \geq \int_1^n \frac{dx}{\sqrt{x+1}} \quad (n \geq 2)$$

and deduce that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

diverges.