G02114

OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2003

SECOND ENGINEERING EXAMINATIONS

MA250

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Time allowed: *Three* hours.

Attempt *five* questions, at least *two* from each section.
Please use separate answer books for each section.

SECTION A

1. (a) Show that the function $u = e^x \cos y + 2xy$ is harmonic and find a harmonic conjugate v of u. Express the function $u + vi$ as a differentiable function of z.

(b) Find the real and imaginary parts of the function $w = \cos z$ and verify that they satisfy the Cauchy-Riemann equations.

2. (a) Derive the Taylor series of the functions e^z, $\cos z$ and $\sin z$ at $z_0 = 0$ and hence prove the identity

$$e^{iz} = \cos z + i \sin z.$$

(b) Let C be the path $z = e^{it}$, $0 \leq t \leq 2\pi$. Prove from first principles that

$$\int_C \frac{dz}{z} = 2\pi i. \tag{1}$$

State Cauchy's Integral Theorem (without proof) and deduce from it and from equation (1) above that, for any z_0 such that $|z_0| < 1$,

$$\int_C \frac{dz}{z - z_0} = 2\pi i.$$

3. Assume Cauchy's formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(Z)\, dz}{Z - z},$$

where C is a circle containing the point z and $f(z)$ is differentiable in a region containing C. Let z_0 be the centre of C. For $n = 0, 1, 2, 3, \cdots$, derive the formulae

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(Z)\, dZ}{(Z - z_0)^{n+1}}$$

and deduce the inequalities

$$|f^{(n)}(z_0)| \le \frac{Mn!}{r^n},$$

where M is a number such that $|f(Z)| \le M$ for all Z on the circle with centre z_0 and radius r.

4. (a) Derive the Laurent expansion of the function $f(z) = \dfrac{1}{z^2 - 1}$ about the point $z = 1$ and hence find the residue of this function at $z = 1$.

(b) Explain what is meant by saying that the function $w = f(z)$ has a pole of order three at $z = z_0$.

Let

$$f(z) = \frac{1}{z^2 \sin 2z}.$$

Show that $f(z)$ has a pole or order three at $z = 0$. Show that $z = \pi/2$ is a simple pole of $f(z)$ and that the residue of $f(z)$ at $z = \pi/2$ is $\dfrac{-2}{\pi^2}$.

5. Use the method of residues to evaluate *two* of the following integrals:

$$(i)\ \int_0^{2\pi} \frac{d\theta}{13 + 5\cos\theta}, \quad (ii)\ \int_{-\infty}^{\infty} \frac{dx}{x^4 + 5x^2 + 6}. \quad (iii)\ \int_0^{\infty} \frac{x \sin 2x\, dx}{x^2 + 4}.$$

SECTION B

6. (a) Let b_u be the projection of a vector b onto a subspace U of R^n. Let u be any vector in U. Explain why $(b-b_u)\cdot(b_u-u)=0$ and hence show that $\|b-u\|^2 \geq \|b-b_u\|^2$.

(b) (i) Use the Gram-Schmidt process to find an orthogonal basis of the subspace U of R^4 spanned by the vectors

$$(1,0,1,0), \qquad (1,-1,0,1), \qquad (1,1,1,0).$$

(ii) Hence, or otherwise, find the projection of the vector $v=(2,0,-1,1)$ onto U.

7. (a) Let $\{u_1,\ldots\ldots,u_k\}$ be a basis of the subspace U of R^n. Write down the formula for P the projection matrix which projects a vector $v\in R^n$ onto the subspace U. Show that, in particular, if $\{u_i\}$ is an orthonormal basis for U then

$$P = AA^T$$

where A is the $n\times k$ matrix having $u_1,\ldots\ldots,u_k$ as columns.

(b) Find the least squares approximate solution to the following over-determined system of equations:

$$\begin{aligned} x+y+z &= 0 \\ -x \quad\; +z &= 1 \\ x-y \quad\; &= -1 \\ y-z &= -2 \end{aligned}$$

8. (a) Let A be a real symmetric matrix. Prove that the eigenvectors corresponding to different eigenvalues are orthogonal to each other.

(b) Consider the symmetric matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & -2 & 2 \end{pmatrix}$$

Verify that 1 is an eigenvalue of A, and find the other eigenvalues.
Determine the eigenvectors corresponding to each eigenvalue, and write down an orthogonal matrix P and a diagonal matrix D such that

$$P^t AP = D$$

9. (a) Apply the Gram-Schmidt process to the functions $1, x, x^2$ to find polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$ with degrees 0, 1, 2 respectively, which are orthogonal with respect to the inner product

$$f \cdot g = \int_{-1}^{1} f(x)g(x)dx$$

(b) Use the polynomials $p_0(x)$, $p_1(x)$, $p_2(x)$ to find a quadratic polynomial which approximates the function

$$f(x) = 2 + |x| = \begin{cases} 2 - x & -1 \leq x < 0 \\ 2 + x & 0 \leq x \leq 1 \end{cases}$$