GX2115

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2003

SECOND UNIVERSITY EXAMINATION

MATHEMATICS [MA 283 (Linear Algebra)]

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Time allowed: **Two** hours.
Full marks for 3 correct solutions.

Q1. (a) Let $\mathbf{V}$ be a finite dimensional vector space over the real numbers $\mathbf{R}$. Define the terms (i) *spanning set* S, (ii) *linearly independent set* I and (iii) *basis* B where S, I and B are finite subsets of $\mathbf{V}$.

(b) Consider the 5×5 matrix A below:

$$\begin{pmatrix} -1 & 2 & 1 & -1 & 1 \\ -2 & 3 & 0 & 1 & -1 \\ -3 & 5 & 1 & 0 & 0 \\ -3 & 6 & -6 & 0 & -6 \\ -4 & 5 & -2 & 5 & -5 \end{pmatrix}$$

Determine the *column rank* of A. Find also a basis for the *null space* of A. Explain why the *row rank* of A is equal to its column rank.

(c) For each positive integer k let $f_k : \mathbf{R} \to \mathbf{R}$ be defined by:

$$f_k(x) := \exp r_k x$$

where each $r_k \in \mathbf{R}$. Prove that $\{f_1, f_2, \ldots, f_n\}$ is linearly independent if and only if $r_1, r_2, \ldots, r_n$ are distinct.

Q2. (a) Let f be a linear mapping from the finite dimensional vector space $\mathbf{V}$ into the finite dimensional vector space $\mathbf{W}$. Define $ker(f)$, the *kernel* and $Im(f)$, the *image* of the linear map f.
Prove that $ker(f)$ is a subspace of $\mathbf{V}$ and that $Im(f)$ is a subspace of $\mathbf{W}$.

(b) Let $f : \mathbf{R}^4 \mapsto \mathbf{R}^3$ be defined by $f(a,b,c,d) = (a+b, b-c, a+d)$. Write down the 3×4 matrix corresponding to this mapping f with respect to the natural (usual) basis, and hence or otherwise find a basis for $Im\ (f)$ and $ker(f)$.

(c) Expand the basis of $ker(f)$ in part (b) to obtain a basis for $\mathbf{R}^4$.

Q3. (a) Let V be a vector space of finite dimension $n \geq 1$ Let $f : V \to V$ be a linear mapping.
Prove that $ker(f) = ker(f^2) \Longleftrightarrow Im(f) = Im(f^2)$.

(b) Prove that if $ker(f) = ker(f^2)$ then $ker(f) \cap Im(f) = 0_V$.

(c) Prove that if $ker(f) \cap Im(f) = 0_V$, then $Im(f) = Im(f^2)$.

Q4. (a) Let A be an $n \times n$ matrix over the real numbers $\mathbf{R}$. Prove that there exists an invertible matrix E such that $E^{-1}AE$ is diagonal if and only if A has n linearly independent eigenvectors.

(b) Prove that if A is an $n \times n$ matrix over $\mathbf{R}$ with n distinct real eigenvalues then A is diagonalizable (ie there exists an invertible matrix E such that $E^{-1}AE$ is diagonal).

(c) For the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

find a matrix E such that

$$E^{-1}AE = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(You should calculate E^{-1} in verifying the above.)

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Q5. (a) Describe the **Gram–Schmidt** process for converting a basis for a finite dimensional Euclidean vector space $\mathbf{V}$ into an *orthonormal* basis for $\mathbf{V}$.

(b) Let S be a real symmetric matrix. Prove that the eigenvectors corresponding to distinct eigenvalues of S are orthogonal.

(c) Let

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix}.$$

Find an orthogonal matrix O such that

$$O^{\mathrm{T}} S O = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$