OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2003

SECOND ARTS and SCIENCE EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS [MA287 — COMPLEX ANALYSIS]

HONOURS

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Time allowed: *Two* hours.
Answer *three* questions

1. (a) Give the definition of the exponential function, e^z and show that it takes every nonzero value. Sketch the image under the mapping $z \mapsto e^z$ of the region

$$\{z \in \mathbb{C} : \pi/4 \leq \operatorname{Im} z \leq \pi/2\}.$$

(b) Give the definition of the complex logarithm function, $\operatorname{Log} z$, and show that this function is discontinuous at every point on the positive real axis.

(c) Give the definition of the functions $\sinh z$ and $\cosh z$ and show that they satisfy $\cosh^2 z - \sinh^2 z = 1$ for every $z \in \mathbb{C}$. Find all the zeros of $\cosh z$.

2. (a) Show that the real and imaginary parts of a differentiable function satisfy the Cauchy-Riemann equations. Find a differentiable function whose real part is

$$u(x, y) = x^4 + 6x^2y - 2y^3 - 6x^2y^2 + y^4.$$

(b) What is a *Möbius transformation*? Show that the function $w = 1/z$ maps a circle or line in the z-plane to a circle or line in the w-plane. Explain briefly why this property holds for every Möbius transformation. Find a Möbius transformation that maps the left half-plane to the region inside the circle with centre i and radius 3.

p.t.o.

3. (a) Give the definition of the integral $\int_\gamma f(z)\,dz$, where γ is a smooth path in the complex plane and state the *Fundamental Theorem* for this integral. Evaluate the integral

$$\int_\gamma \frac{dz}{(z-a)^n},$$

where n is a natural number and γ is the circle with centre a and radius r, traversed anticlockwise. You should distinguish between the cases $n = 1$ and $n > 1$.

(b) Explain why it follows from the Fundamental Theorem that

$$\int_\gamma \frac{dz}{z} = 0$$

if γ is any circle for which the origin lies outside γ.

(c) Answer **either** part (I) **or** part (II):

(I) State and prove *Cauchy's Theorem* for triangular paths.

(II) If f is differentiable at every point in a star-shaped domain O, prove that f has a primitive in O.

4. (a) State and prove the *Cauchy Integral Formula.* Use it to evaluate the integral

$$\int_\gamma \frac{\cos z}{z^2 - 6z + 5}\,dz,$$

where γ is the circle $|z| = 4$.

(b) Define the terms *pole* and *residue* and state the *Residue Theorem.* Find the residues of the function

$$f(z) = \frac{\cos z}{(z-i)z^3}$$

at each of its poles. Hence evaluate the integral

$$\int_\gamma \frac{\cos z}{(z-i)z^3}\,dz,$$

where γ is the circle with centre -1 and radius 2.