GD1971

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER 1 EXAMINATIONS 2003-2004

SECOND ENGINEERING EXAMINATION

MATHEMATICS [MA250]

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Time allowed: **Three** hours.
Full marks for five correct solutions.
Answer **not more than three** questions from each section.
Please use **separate answer books** for each section.

SECTION A

1. (a) Determine whether the series

$$\sum_{k=1}^{\infty} \frac{sin^2 k}{(k+1)^{5/4}}$$

converges or not.

(b) For which values of x does the power series

$$\sum_{n=0}^{\infty} n^3 (x+1)^n$$

converge?

(c) Define the n-th order Taylor polynomial $P_n(x)$ of a function at a point $x = a$. Determine the second order Taylor polynomial at $x = 1$ for the function $g(x) = cos(e^x)$.

p.t.o.

2. (a) Determine the tangent plane to the graph of the function $f(x, y) = x^2e^y$ at the point $(1, 0, 1)$.

   (b) Is there a value d such that the tangent plane to the graph of the function $f(x, y) = x^2e^y$ and the tangent plane to the surface given by $x^4/2 + xy + z^d = 3/2$ at the point $(1, 0, 1)$ are orthogonal to each other? If this is the case, determine all such d.

3. (a) Determine how the partial differential equation

$$-2x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = xy^4$$

   is transformed when the new variables $u = xy^2$ and $v = y$ are introduced. (The obtained expression should not contain x or y.)

   (b) Determine a function $f(x, y)$ which is a solution to the differential equation from part (a) and also satisfies $f(1, y) = 0$.

   (c) Let f be an arbitrary sufficiently differentiable function. Express $\frac{\partial^2 f}{\partial x \partial y}$ in the variables u and v introduced in part (a).

4. (a) Determine the points on the curve $4(x-y)^2+(x+y)^2 = 1$ for which the distance to the origin is maximal and minimal, respectively.

   (b) Evaluate the integral

$$\int\int_T xy\, dy\, dx$$

   where T is the region bounded by the inequalities $y \geq 0$, $y \leq 3-x$, $y \leq x - 1$.

5. (a) Consider the function $f(x, y) = (x^2 - y^2)e^{-2x+y}$. Determine all critical points of f in $\mathbb{R}^2$ and for each critical point, determine if it is a local maximum, local minimum or a saddle point.

   (b) Evaluate the integral

$$\int\int_D \frac{x^2}{x^2 + y^2} dx\, dy$$

   where D is the region bounded by $1 \leq x^2 + y^2 \leq 4$.

SECTION B

6. (a) What is a *subspace* of $\mathbb{R}^n$? Determine whether the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$.

(i) $\{(x, y, z) \mid x + y = 2z\}$

(ii) $\{(x, y, z) \mid x + y > 2z\}$

(b) What is the *span* of a set of vectors in $\mathbb{R}^n$? Prove that the span of any subset of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

(c) Find a spanning set for the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$, where $A = \begin{pmatrix} 3 & 1 & 1 \\ 6 & 2 & 2 \\ -9 & -3 & -3 \end{pmatrix}$, $\mathbf{x} = (x_1, x_2, x_3)^{\mathsf{T}}$.

7. (a) Determine whether or not the vectors $(1, 1, 1, 2)$, $(3, 0, -1, 4)$ and $(2, 5, 5, 1)$ are linearly independent in $\mathbb{R}^4$.

(b) Let W be a subspace of $\mathbb{R}^n$. Define: *basis* of W, *dimension* of W. Explain why $\dim(W) \leq n$.

(c) Let $W \subset \mathbb{R}^5$ be the span of

$$\{(1, -1, 0, 2, 1),\ (2, 1, -2, 0, 0),\ (0, -3, 2, 4, 2),\ (3, 3, -4, -2, -1),\ (2, 4, 1, 0, 1),\ (5, 7, -3, -2, 0)\}.$$

Find a basis of W.

8. (a) Determine whether the following functions are linear transformations.

(i) $T\colon \mathbb{R} \to \mathbb{R},\ T(x) = \tan x$

(ii) $T\colon \mathbb{R}^2 \to \mathbb{R},\ T(x, y) = x - 2y$

(b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + y, 3z, z - 2x).$$

(i) Find the standard matrix representation of T, and compute $T(1, -1, 7)$.

(ii) Find the kernel of T.

(iii) Prove that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a basis of $\mathbb{R}^3$ (here $\mathbf{e}_i$ is the vector with 1 in position i and zeroes elsewhere).

p.t.o.

9. (a) Find all eigenvalues of the matrix

$$A = \begin{pmatrix} 6 & 3 & -3 \\ -2 & -1 & 2 \\ 16 & 8 & -7 \end{pmatrix}.$$

(b) Find three linearly independent eigenvectors for A.

(c) Using parts (a) and (b), compute A^{25}.