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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER I EXAMINATIONS 2003-2004

MA 235 – PROBABILITY AND STATISTICS

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Time Allowed: **Two** hours
Answer the ten questions in PART A (30 marks)
and
two of the questions in PART B (35 marks each).

PART A

[*Multiple choice.* [30 *marks*]] In each of questions **A1**. through **A10**. below, **write down one** choice of answer, and ensure that you write **both the number of the answer and the answer itself**. For example, if in **A1**. below you think **(a)** is the answer, you would write in your answer book: A1. (a) $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$.

A1. If five cards will be picked at random from a pack of 52 cards, **what** is the probability that we will obtain three of a kind (that is, five face values of the form (x,x,x,y,z) where x,y and z are distinct)?

(a) $\binom{13}{1}\binom{12}{2}\binom{4}{3}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$ **(b)** $\binom{13}{1}\binom{12}{1}\binom{4}{1}/\binom{52}{5}$
(c) $\binom{13}{4}\binom{12}{1}\binom{4}{3}\binom{4}{1}/\binom{52}{5}$ **(d)** $\binom{13}{2}\binom{11}{1}\binom{4}{2}\binom{4}{2}\binom{4}{1}/\binom{52}{5}$
(e) $\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}\binom{4}{1}\binom{4}{1}/\binom{52}{5}$ **(f)** $\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}/\binom{52}{5}$.

A2. The distribution function $F(x) = P(X \leq x), -\infty < x < \infty$ of a random variable X is given by

$$F(x) = \begin{cases} 1 - \frac{9}{x^2}, & \text{for } x > 3 \\ 0, & \text{elsewhere.} \end{cases}$$

What is $P(X \leq 4)$?

(a) $\frac{7}{16}$ **(b)** $\frac{9}{16}$ **(c)** 0 **(d)** 1 **(e)** $\frac{3}{4}$ **(f)** none of these.

A3. In how many ways can 10 bags of Taytos be distributed among 3 students?

(a) $\binom{12}{10}$ **(b)** 10^3 **(c)** 3^{10} **(d)** $\binom{12}{3}$ **(e)** $\binom{10}{3}$ **(f)** none of these.

A4. **In how many** ways can the 9 letters of the word MARMALADE be arranged in a row without changing the order of the 4 vowels A, A, A, E ?

(a) $\frac{1}{5!} \times \frac{9!}{2!3!1!1!1!1!}$ **(b)** $\frac{1}{4} \times \frac{9!}{2!3!1!1!1!1!}$ **(c)** $\frac{1}{3!} \times \frac{9!}{2!3!1!1!1!1!}$ **(d)** $\frac{1}{2} \times \frac{9!}{2!3!1!1!1!1!}$
(e) $\frac{1}{3} \times \frac{9!}{2!3!1!1!1!1!}$ **(f)** $\frac{1}{4!} \times \frac{9!}{2!3!1!1!1!1!}$ **(g)** $\frac{9!}{2!3!1!1!1!1!}$ **(h)** none of these.

A5. Assume that the probability of a male birth is 0.5 and that births are independent of each other. A family with 4 children is selected at random. Given that at least 3 of the children are males, what is the probability that all 4 are males?

(a) $\frac{1}{6}$ **(b)** $\frac{1}{5}$ **(c)** $\frac{2}{5}$ **(d)** $\frac{4}{7}$ **(e)** $\frac{3}{5}$ **(f)** $\frac{4}{5}$.

A6. A and B are events satisfying $P(A) = P(A|B) = 0.6$. If $\overline{A}$ denotes the complement of A, then **which** of the following statements is true?

(a) A and B are disjoint (i.e. mutually exclusive), and $P(\overline{A}|B) = 0.6$ **(b)** A and B are disjoint, and $P(\overline{A}|B) = 0.4$
(c) A and B are disjoint, and $P(\overline{A}|B) = 0.5$ **(d)** A and B are not disjoint, and $P(\overline{A}|B) = 0.25$
(e) A and B are not disjoint, and $P(\overline{A}|B) = 0.4$ **(f)** A and B are not disjoint, and $P(\overline{A}|B) = \frac{2}{3}$.

A7. Suppose that 9 students will be distributed at random into 3 classes in such a way that each class will get 3 students. If there are 3 whiz kids among the 9 students, **what** is the probability that each class gets one?

(a) $\frac{1}{3}$ **(b)** $\frac{2}{9}$ **(c)** $\frac{5}{72}$ **(d)** $\frac{1}{6}$ **(e)** $\frac{3}{56}$ **(f)** $\frac{9}{28}$

A8. The following randomized response technique can be used by an interviewer to elicit answers to sensitive questions. Suppose that we want to estimate $P(I)$, the proportion of executives of Irish financial companies who have offshore bank accounts. We construct 100 flash cards, write "I have an offshore bank account" on 60 of the cards, and write "I do not have an offshore bank account" on the remaining 40 cards. Each executive, in a random sample of 50 executives interviewed, chooses a card at random and, without divulging the question on the card to the interviewer, truthfully responds "yes" or "no" to the statement on the card. [*Note*: An executive will say "yes" if he/she does have an offshore account and the card he/she chooses has "I have an offshore bank account" written on it, and he/she will also say "yes" if he/she does not have an offshore account and the card he/she chooses has "I do not have an offshore bank account" written on it.] Suppose that 20 of the 50 executives respond "yes". **Which** of the equations below do we solve to obtain an estimate p of $P(I)$?

(a) $\frac{70}{100} = \frac{40}{100}p + \frac{60}{100}[1-p]$ **(b)** $\frac{20}{50} = \frac{40}{100}p + \frac{60}{100}[1-p]$ **(c)** $\frac{20}{50} = \frac{60}{100}p + \frac{40}{100}[1-p]$.

A9. An art dealer has 2 paintings and feels that the probabilities are 0.5, 0.3 and 0.2 that 0,1 or 2 of them, respectively, are forgeries. She selects one of the paintings at random and sends it away for authentication. If it transpires that this painting is a forgery, **what** is the (conditional) probability that the other painting is also a forgery?

(a) 0 **(b)** $\frac{1}{4}$ **(c)** $\frac{1}{2}$ **(d)** $\frac{2}{3}$ **(e)** $\frac{3}{4}$ **(f)** 1.

A10. Suppose that X_1 and X_2 are independent random variables satisfying $E(X_1) = 2$, $E(X_2) = 2$, $Var(X_1) = 1$, $Var(X_2) = 1$. **What** is $E(X_1^2 - X_1X_2)$?

(a) 0 **(b)** 1 **(c)** 1 **(d)** 1 **(e)** 1 **(f)** 5.

PART B

B1.

(a) [11 *marks*] Prove **exactly one** of the two results (i) and (ii) below.

(i) Let X represent the number of "successes" we will obtain when n items are taken without replacement from a population that consists of N items, of which a are of one kind (success) and the remaining $N-a$ are of a second kind (failure). **Show** that for each fixed $x = 0,1,2,\ldots,\min(a,n)$, the hypergeometric probability $P(X=x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$ converges to the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ as N tends to ∞ and a tends to ∞ in such as way that the proportion of successes in the population $\frac{a}{N}$ tends to θ.

OR

(ii) **Show** that for each fixed $x = 0,1,2,\ldots,n$, the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ converges to the Poisson probability $e^{-\lambda}\frac{\lambda^x}{x!}$ as n tends to ∞ and θ tends to 0 in such as way that $\lambda = n\theta$ remains fixed.

(b) The number X of emails sent by a student during any time period $(0,t)$ during a random Monday has a Poisson distribution with, on average, 2 emails per hour. Thus the density function of X is

$$P(X = x \text{ in time period of length } t \text{ hrs.}) = e^{-2t}\frac{(2t)^x}{x!}, \quad x = 0,1,2,\ldots.$$

(i) [4 *marks*] **Write down** the probability of no email being sent in the first two hours of Monday.

(ii) [4 *marks*] Use the fact that the time T until the first email is sent by the student on a random day has the exponential density $f_T(t) = \begin{cases} 2e^{-2t}, & \text{for } t > 0 \\ 0, & \text{elsewhere} \end{cases}$ to **re-calculate** the probability of no email being sent in the first two hours of Monday.

(c) [4 *marks*] There are 10,000 students at a certain university, of whom 6,000 are males and 4,000 are females. Let $K = \dfrac{\binom{6,000}{300}\binom{4,000}{200}}{\binom{10,000}{500}}$. **Write down** which one of (i), (ii), (iii) and (iv) below represents the percentage error if we us the binomial distribution in place of the hypergeometric distribution to approximate the probability that exactly 300 of 500 randomly selected students are males.

(i) $\dfrac{K - \binom{10,000}{500}(0.6)^{6,000}(0.4)^{4,000}}{K} \times 100\%$ (ii) $\dfrac{K - \binom{500}{300}(0.6)^{300}(0.4)^{200}}{K} \times 100\%$

(iii) $\dfrac{K - \binom{10,000}{500}(0.6)^{300}(0.4)^{200}}{K} \times 100\%$ (iv) $\dfrac{\dfrac{\binom{6,000}{300}\binom{4,000}{200}}{\binom{10,000}{6,000}} - \binom{10,000}{500}(0.6)^{300}(0.4)^{200}}{\dfrac{\binom{6,000}{300}\binom{4,000}{200}}{\binom{10,000}{6,000}}} \times 100\%.$

(d) [4 *marks*] In a classroom, there are 5 students, 3 of whom are males. If 3 students will be selected at random, let a denote the probability that exactly 2 of them are males when sampling is without replacement, and let b be the probability that exactly two of them are males when sampling is with replacement. **Write down** which one of (i), (ii), (iii) and (iv) below gives the correct values for a and b ?

(i) $a = \dfrac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}}$ and $b = \binom{3}{2}(\frac{3}{5})^2(\frac{2}{5})^1$ (ii) $a = \binom{3}{2}(\frac{3}{5})^2(\frac{2}{5})^1$ and $b = \dfrac{\binom{3}{2}\binom{2}{1}}{\binom{5}{3}}$

(iii) $a = \binom{5}{2}(\frac{4}{12})^2(\frac{8}{12})^3$ and $b = \binom{5}{2}(\frac{4}{12})^2(\frac{4}{12})^2(\frac{1}{12})$ (iv) $a = \frac{1}{2}$, $b = \frac{1}{2}$.

(e) [4 *marks*] In a list of 9 stock portfolios, 4 are value-weighted, 3 are price-weighted, and 2 are equal-weighted. If 5 portfolios are randomly selected without replacement from the list, let $a = P$(exactly 2 are value-weighted) and let $b = P$(exactly 2 are value-weighted and 2 are price-weighted). **Write down** which one of (i), (ii), (iii) and (iv) below gives precisely correct values for a and b?

(i) $a = \dfrac{\binom{4}{2}\binom{5}{3}}{\binom{9}{5}}$ and $b = \dfrac{\binom{4}{2}\binom{3}{2}\binom{2}{1}}{\binom{9}{5}}$ (ii) $a = \binom{9}{5}(\frac{4}{9})^2(\frac{5}{9})^3$ and $b = \binom{9}{5}(\frac{4}{9})^2(\frac{4}{9})^2(\frac{5}{9}$

(iii) $a = \binom{5}{2}(\frac{4}{9})^2(\frac{5}{9})^3$ and $b = \binom{5}{2}(\frac{4}{9})^2(\frac{4}{9})^2(\frac{5}{9})$ (iv) none of these.

(f) [4 *marks*] Suppose that 12 of your friends will independently purchase one lottery ticket for the next draw of the National Lottery (in which 6 numbers are drawn from $\{1, 2, \ldots, 42\}$, and then a bonus ball is drawn). Let X = number of your five friends who match 'three and the bonus'. **Which one** of (i), (ii), (iii), (iv) and (v) below gives the correct value for $E(X)$, the expected number of your friends who match 'three and the bonus'?

(i) $\dfrac{1}{\binom{42}{6}}$ (ii) $\dfrac{12}{\binom{42}{6}}$ (iii) $\binom{12}{6} \times \left[1 - \dfrac{1}{\binom{42}{6}}\right] \times \dfrac{1}{\left[\binom{42}{6}\right]^4}$

(iv) $\dfrac{1}{\left[\binom{42}{6}\right]^{12}}$ (v) $\left[1 - \dfrac{1}{\binom{42}{6}}\right]^{12}$.

B2.

(a) [10 *marks*] Let X have the uniform $(0, 1)$ density function $f_X(x) = \begin{cases} 1, 0 < x < 1 \\ 0, \text{elsewhere} \end{cases}$. **Find** the density function of $Y = -\theta \ln(1 - X)$ for a constant $\theta > 0$.

(b) Modern theories concerning the relationship between risk and return, along with empirical evidence, suggest that the average rate of return for stocks listed in the over-the-counter (OTC) market exceed those of the more mature companies in the New York Stock Exchange (NYSE). Suppose that the annual return X_1 (in %) on OTC stocks and the annual return X_2 (in %) on NYSE stocks are independent and that $X_1 \sim N(12, 16)$ and $X_2 \sim N(9, 9)$.

(i) [5 *marks*] **What** proportion of OTC stocks have annual returns in excess of 20%?

Note: If $Z \sim N(0,1)$, then $P(Z > \frac{3}{5}) = 0.2743$, $P(Z > 1) = 0.1587$, $P(Z > 1.96) = 0.025$, $P(Z > 2) = 0.0228$.

(ii) [10 *marks*] **What** is the probability that a randomly selected OTC stock will outperform a randomly selected NYSE stock? That is, **find** $P(X_1 > X_2)$. [*Hint*: You will need one of the Z probabilities from the previous part. Also recall the following important result about the distribution of linear combinations of independent normally distributed random variables: If $X_1, X_2, \ldots, X_n$ are independent with

$X_i \sim N(\mu_i, \sigma_i^2), i = 1, 2, \ldots, n,$ then for any constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\sum_{i=1}^n \alpha_i X_i \sim N\left(\sum_{i=1}^n \alpha_i \mu_i, \sum_{i=1}^n \alpha_i^2 \sigma_i^2\right).]$$

(iii) [10 *marks*] Suppose that you will invest α_1 Euro from your funds in OTC stocks and α_2 Euro in NYSE stocks. Let $Y = \alpha_1 X_1 + \alpha_2 X_2$ be your profit (in Euro) next year. **Find** the values of α_1 and α_2 if it is desired that the mean profit, $E(Y)$, be 36 Euro and that the variance of the profit, $Var(Y)$, be minimised.

B3.

(a)

(i) [5 *marks*] **Prove** Chebychev's Inequality: if X has finite variance σ^2, then for any $k > 0$,

$$P(|X - E(X)| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

Hint: You may use, if you wish, the following inequality of Markov: If $u(X)$ is a nonnegative function of a random variable X and if $E[u(X)]$ exists, then for each constant $c > 0$,

$$P(u(X) \geq c) \leq E[u(X)]/c.$$

(ii) [5 *marks*] Let X denote the number of heads we will obtain in 900 flips of a fair coin. Use Chebychev's Inequality to **place** a lower bound on the probability $P(405 < X < 495)$.

(iii) [5 *marks*] **Calculate** the approximate value of $P(405 < X < 495)$ in (ii) using the normal approximation to the binomial distribution. *Note*: If $Z \sim N(0,1)$, then $P(Z > 1) = 0.1587$, $P(Z > 2) = 0.0228$, $P(Z > 3) = 0.0013$.

(b) Tom and Mary have decided to meet at a certain nightclub sometime between midnight (time 0) and 2 am (time 2). Assume that their respective arrival times X and Y are independent and each is uniformly distributed on the interval (0, 2). Hence the joint density of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}, & 0 < x < 2,\ 0 < y < 2 \\ 0, & \text{elsewhere}. \end{cases}$$

They have agreed that whoever arrives first will wait for at most 1/2 hour for the arrival of the other.

(i) [10 *marks*] **Find** the probability that they meet by integrating the joint density $f_{X,Y}(x, y)$ over an appropriate region. Ensure that you draw a rough sketch of this region.

(ii) [10 *marks*] *Accept* that the density function of the random variable $U = X - Y$ is

$$f_U(u) = \begin{cases} \frac{1}{2} + \frac{u}{4}, & -2 < u < 0 \\ \frac{1}{2} - \frac{u}{4}, & 0 < u < 2 \\ 0, & \text{elsewhere}. \end{cases}$$

Use the density $f_U(u)$ to **re-calculate** the probability that Tom and Mary meet.