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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER 1 EXAMINATIONS 2003/2004

THIRD UNIVERSITY EXAMINATION

MATHEMATICS — [MA313]

LINEAR ALGEBRA I

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Time allowed: *Two* hours.
Third Arts Mathematical Studies: Full marks for *four* questions.
All other students: Full marks for *three* questions.

1. (a) What is meant by a *basis* for a vector space?
Determine with proof whether or not

$$\{(1,1,-2),(2,1,0),(3,2,-3)\}$$

is a basis for $\mathbb{R}^3$.
Determine whether or not

$$\{x, 2+x^2, 1+x+x^2\}$$

is linearly independent in $\mathbb{R}[x]$.

(b) Let

$$X = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \right\}$$

Show that X is linearly dependent. Find a basis for the subspace generated by X.

(c) What is meant by a *subspace* of a vector space? Suppose U, W are subspaces of of vector space V. Prove that $U \cap W$ is a subspace of V.
Show by an example that it is not necessary that $U \cup W$ be a subspace.

p.t.o.

2. (a) Let U be the solution space in $\mathbb{R}^4$ of

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & - & x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 0 \end{array}$$

and let V be the solution space in $\mathbb{R}^4$ of

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \end{array}$$

Find bases for U, V and $U+V$.

(b) Let $f : V \to W$ be a linear transformation of vector spaces. Explain *kernel* of f, $\ker f$, and *image* of f, $Im\, f$. Show that $\ker f$ is a subspace of V. Show also that $f(0_V) = 0_W$.

(c) Find the coordinate vector of $(x-1)^2$ with respect to the (ordered) basis $\{x^2, x+1, 1\}$ for $\mathbb{R}_2[x]$ (the vector space of polynomials of degree ≤ 2 over $\mathbb{R}$).

3. (a) Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

be the matrix of $f : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$. Find the matrix of f relative to the the basis $W = \{\mathbf{u_1} = (1,1,2)^t, \mathbf{u_2} = (2,0,1)^t, \mathbf{u_3} = (1,1,1)^t\}$ for $\mathbb{R}^3$ and the basis $V = \{\mathbf{v_1} = (1,1)^t, \mathbf{v_2} = (0,1)^t\}$ for $\mathbb{R}^2$.

(b) The matrix A below is the matrix of a linear transformation $f : \mathbb{R}^4 \to \mathbb{R}^3$ relative to the natural (standard) bases for $\mathbb{R}^4$ and $\mathbb{R}^3$. Write out the transformation in the form $f : \mathbb{R}^4 \to \mathbb{R}^3$ explicitly.

Find bases for $\ker f$ and $\mathrm{Im}\ f$ and verify that $\dim \ker f + \dim\ \mathrm{Im}\ f = 4$.

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & -2 & 1 & -1 \end{pmatrix}$$

4. (a) Explain *eigenvalue* and *eigenvector* of a square matrix A. Suppose there exists an invertible matrix E and a matrix B such that $E^{-1}AE = B$. Prove that B and A have the same eigenvalues.

If B is a diagonal matrix what does this say about the eigenvalues of A? Show that the columns of E, in this case, are eigenvectors of A.

(b) Let

$$A = \begin{pmatrix} -1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$$

Show that the eigenvalues of A are $\lambda = -2, -2, 4$. Find if it is possible to obtain a nonsingular E and a diagonal D such that $D = E^{-1}AE$ and if so find such an E.