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NATIONAL UNIVERSITY OF IRELAND, GALWAY

WINTER EXAMINATIONS 2003

B.A. and B.Sc. Degree Examination
H.Dip. Applied Science

MA341 - Metric Spaces

Dr Dave Johnson
Professor T. Hurley
Dr Graham Ellis

Time allowed: *two* hours.
Answer *three* questions.

1. (a) Define the terms *metric space* and *contraction mapping.*

   (b) Let $C[0,1]$ be the metric space of continuous functions $x\colon[0,1]\to\mathbf{R}$ with metric $d(x,y)=\max\{|x(t)-y(t)| : t\in[0,1]\}$. Prove that $g(x)(t)=2+\int_0^t ux(u)\,du$ is a contraction mapping on $C[0,1]$.

   (c) Let $x_0(t)=2$ for all $t\in[0,1]$ and set $x_{i+1}=g(x_i)$ with $g(x)$ as above. Determine the functions $x_1(t)$, $x_2(t)$ and, by solving the appropriate differential equation, determine also the function $x(t)$ to which the sequence $x_0(t), x_1(t), \cdots$ converges.

   (d) Define a *Cauchy sequence* and then decide whether

   $$y_n(t)=t^n$$

   specifies such a sequence in $C[0,1]$. Justify your answer.

2. (a) Use the Mean Value Theorem to prove that $g(x)=\frac{1}{7}(x^5+x+1)$ is a contraction mapping on $[-1,1]$. Then use the Contraction Mapping Theorem to show that the equation $x^5-6x+1=0$ has a unique solution in $[-1,1]$.

   (b) Use the Implicit Function Theorem to prove that the equation

   $$2xt+\sin(x)=e^{t^2}$$

   defines a unique continuous function on $[1,\infty)$.

   (c) Prove that any *compact* metric space is *complete.* Include definitions of the two italicized terms.

3. (a) State the Contraction Mapping Theorem and the Implicit Function Theorem.

   (b) Prove *one* of the above two theorems.

   (c) Prove that $\mathbf{R}^2$ is complete with respect to the Euclidean metric. (You may assume the completeness of $\mathbf{R}$.)

   (d) Give an example of a function $f: [0, 1) \to [0, 1)$ which has no fixed point and which is a contraction with respect to the usual metric.

4. (a) Let S^2 denote the unit sphere in $\mathbf{R}^3$ endowed with the "shortest path" metric. Let Δ denote the area of a geodesic triangle on S^2 with interior angles α, β, γ. Prove that

$$\alpha + \beta + \gamma = \pi + \Delta.$$

   (b) Define the *hyperbolic distance* $d_{\mathbf{H}}(z, z')$ between two complex numbers in the upper-half plane, and prove that

$$d_{\mathbf{H}}(a\mathbf{i}, b\mathbf{i}) = \ln \frac{b}{a}$$

   for real numbers $b > a > 0$ and $\mathbf{i} = \sqrt{-1}$.

   (c) Find a function $T \in PSL(2, \mathbf{R})$ which maps both $-3 + 4\mathbf{i}$ and $3 + 4\mathbf{i}$ to the imaginary axis.

   (d) Calculate the hyperbolic distance between $-3 + 4\mathbf{i}$ and $3 + 4\mathbf{i}$.

5. Let (X, d) be a metric space.

   (a) Prove that all contraction mappings $g : X \to X$ are continuous.

   (b) Show that

$$|d(x, z) - d(y, u)| \leq d(x, y) + d(z, u)$$

   for all $u, x, y, z \in X$. Then show that for any $a \in X$ the function

$$f: X \to \mathbf{R}, x \mapsto d(a, x)$$

   is continuous with respect to the usual metric on $\mathbf{R}$.

   (c) Prove that if a function $f: X \to X$ is continuous then the inverse image $f^{-1}(A)$ of any closed subset A is closed.

   (d) Prove that the composite $h \circ f$ of two continuous functions $f, h: X \to X$ is continuous.