GX 1988

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER I EXAMINATIONS 2003 – 2004

MA 387 – PROBABILITY

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Time Allowed: **Two** hours.

Answer <u>question 1</u> [40 *marks*] and any <u>two</u> of questions 2– 4 [30 *marks each*].

1. [40 *marks*] **Give** *numerical* answers in each of the 11 questions **A**. through **K**. below. Part marks will *not* be allotted in any part, so any work you may exhibit will *not* be examined. *Full marks for* 10 *correct answers.*

A **In how many** ways can 12 hockey players be divided into two teams of 6 players each?

B. There are 2 radar units, operating independently, at an airport. Each of them has a 90% chance of detecting an incoming plane. **What** is the probability that *neither* of 2 planes, arriving independently, will be detected?

C. Each of the 50 states in the U.S. has 2 senators. If 50 senators will be selected at random, **what** is the probability that each state is represented?

D. If five cards will be picked at random from a pack of 52 cards, **what** is the probability that we will obtain four of a kind? [*Note*: A 'four of a kind' hand means five face values of the form (x,x,x,x,y) where x and y are distinct.]

E. If 6 people will be taken at random from the 22 people in a classroom, **what** is the probability that a *particular* group of 3 people will be included in the selection?

F. In any step that he takes, an inebriated student has a 25% chance of stepping forward and a 75% chance of stepping backwards. If he takes 6 steps (independently) in all, **what** is the probability that he will end up 2 steps ahead of his initial position?

G. *Áine*, *Fáinne* and *Gráinne* will take turns flipping a fair coin, with *Áine* being the first to flip. The winner is the one of the three who first obtains a head. **What** is *Áine's* probability of winning?

H. **In how many** ways can the 9 letters of the word MARMALADE be arranged in a row without changing the order of the 4 vowels A, A, A, E ?

I. Suppose that f is an analytic function of three variables: $f = f(x,y,z)$. **How many** potentially different fourth-order partial derivatives of f can be formed? [*Note*: Since the order in which a partial derivative is taken is irrelevant, all that distinguishes one-fourth order partial from another are the number of times the differentiation was done with respect to x, with respect to y, and with respect to z. Thus, for example, $\frac{\delta^4}{\delta x^2 \delta y \delta z} f(x,y,z)$ (i.e. differentiating twice with respect to each of x, y and z in that order), is the same as $\frac{\delta^4}{\delta y \delta x^2 \delta z} f(x,y,z)$, but but need not be the same as $\frac{\delta^4}{\delta x \delta y \delta z^2} f(x,y,z)$ or $\frac{\delta^4}{\delta x^2 \delta y^2} f(x,y,z)$.]

J. Mary, Tom and four other people are are seated at random in six chairs $1,2,3,4,5,6$. **What** is the probability that Mary will occupy chair 3 and Tom will occupy chair 4?

K. The number of mobile phone calls you receive has a Poisson distribution with, on average, 2 calls per hour. **What** is the probability that you receive exactly 1 call in a random 2 hour period?

2.

(a) [5 *marks*] By thinking of a counting problem involving binomial coefficients, prove that $n!!$ is divisible by $((n-1)!!)^n$. *Note*: $m!!$ means $(m!)!$

(b) [10 *marks*] Prove exactly one of the three results (i), (ii) and (iii) below.

(i) Let X represent the number of "successes" we will obtain when n items are taken without replacement from a population that consists of N items, of which a are of one kind (success) and the remaining $N-a$ are of a second kind (failure). **Show** that for each fixed $x = 0, 1, 2, \ldots, \min(a, n)$, the hypergeometric probability $P(X = x) = \frac{\binom{a}{x}\binom{N-a}{n-x}}{\binom{N}{n}}$ converges to the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ as N tends to ∞ and a tends to ∞ in such as way that the proportion of successes in the population $\frac{a}{N}$ tends to θ.

OR (ii)

Show that for each fixed $x = 0, 1, 2, \ldots, n$, the binomial probability $\binom{n}{x}\theta^x(1-\theta)^{n-x}$ converges to the Poisson probability $e^{-\lambda}\frac{\lambda^x}{x!}$ as n tends to ∞ and θ tends to 0 in such as way that $\lambda = n\theta$ remains fixed.

OR (iii)

Let X have the binomial density $P(X = x) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$, $x = 0, 1, \ldots, n$.
Show that $E(X) = n\theta$ and $Var(X) = n\theta(1-\theta)$.

(c) Suppose that the three assumptions underlying a Poisson process hold. That is,

$(A1)$ in a small time interval or amount of space the probability of occurrence of the event of interest is proportional to the size of the interval, i.e. $P(1 \text{ arrival in } [t, t+\Delta t]) = \alpha\Delta t$,

$(A2)$ the probability of two or more occurrences in a small time or space interval is negligible compared with the probability of one occurrence, and

$(A3)$ occurrences in non-overlapping intervals are independent.

Let $f(x, t) = P(x \text{ arrivals in the time interval } (0, t))$.

(i) [5 *marks*] **Show** that $f(x, t+\Delta t) = f(x, t)(1-\alpha\Delta t) + f(x-1, t)(\alpha\Delta t)$.

Hint: Start by copying the following into your answer book:

$$f(x, t+\Delta t) = P(x \text{ successes in the time interval } (0, t+\Delta t]) =$$
$$P(x \text{ arrivals in } (0, t) \text{ and } 0 \text{ arrivals in } [t, t+\Delta t]) +$$
$$P(x-1 \text{ arrivals in } (0, t) \text{ and } 1 \text{ arrival in } [t, t+\Delta t]) +$$
$$P(x-2 \text{ arrivals in } (0, t) \text{ and } 2 \text{ arrivals in } [t, t+\Delta t]) + \ldots .$$

Then note that by $(A2)$ you can ignore all but the first two terms on the right side of this equation.

(ii) [5 *marks*] Hence **show** that $\frac{df(x,t)}{dt} = \alpha[f(x-1, t) - f(x, t)]$. (*)

(iii) [5 *marks*] **Show** that the Poisson density:

$$f(x;t) = e^{-\alpha t}\frac{(\alpha t)^x}{x!}, \quad x = 0, 1, 2, \ldots$$

satisfies the infinite system of differential equations (*) above.

3.

(a) [8 *marks*] A tetrahedron is a three-dimensional figure with four identical sides each of which is an equilateral triangle. Suppose that one face of a tetrahedron is coloured ash, a second face is coloured blue, a third side is coloured crimson, and the fourth side coloured with a combination of all three colours. The tetrahedron will be thrown once. Let A be the event that the face on which it lands contains ash, let B be the event that it contains blue, and let C be the event that it contains crimson. By simply writing down the values of all probabilities that now appear, **show** that, $P(A) = P(A \mid B)$, $P(B) = P(B \mid C)$ and $P(C) = P(C \mid A)$ [so that A, B and C are pairwise independent], and $P(A) \neq P(A \mid B \cap C)$ [so that A, B and C are not mutually independent].

(b) In medical diagnostic testing, the probability that a diseased individual will have a positive test result is called the *sensitivity*, or *true positive rate* (*TPR*) of the test. The probability that a disease-free individual will have a positive test result is called the *false positive rate* (*FPR*) of the test. Suppose now that the proportion of Irish people with cancer is $\frac{1}{200}$, and that a new cancer screening test has a *TPR* of 0.95 and an *FPR* of 0.01.

(i) [5 *marks*] **What** proportion of people who take the test will have a positive result?

(ii) [5 *marks*] Given that a patient has a positive result, **what** is the probability that he/she has cancer?

(c) Suppose that 30% of employees of a large firm take public transport to work.

(i) [3+3 = 6 *marks*] If 20 employees will be taken at random, **what** is the probability that 18 of them take public transport? **What** is the probability that all 20 take public transport given that at least 18 of them take public transport?

(ii) [6 *marks*] Suppose that employees are sampled until we have obtained two who take public transport. **What** is the probability that the tenth employee sampled is the second who takes public transport?

4.

(a) Suppose that n men check in their hats at a cloakroom and that their hats are returned at random (so that the probability is $\frac{1}{n}$ that any particular man gets back his own hat).

(i) [10 *marks*] **Show** that the probability that none of the men gets back his own hat is

$$p_n = 1 - \left\{1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1}\frac{1}{n!}\right\}.$$

(ii) [5 *marks*] Hence **write down** a formula for the probability that exactly r ($\leq n$) of the men get back their own hats. Your answer should involve n, r and p_k where you must give k in terms of n and r.

(b) There are five people in a room, of whom three are lecturers (Pat, Tom and Mary) and the remaining two are students. Three people will be selected at random from the room

(i) [2 *marks*] **Write down** the probability that exactly one of the three lecturers will be selected.

(ii) [6 *marks*] **Calculate** the probability that Pat, Tom and Mary will all be selected given that at least one of them will be selected.

(c) [7 *marks*] Suppose that 6 *distinguishable* objects are placed into 6 cells so that all 6^6 possible arrangements are equally likely. **Find** the number of these that will have exactly one cell empty.