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CHRISTMAS EXAMINATIONS 2003

B.Sc. (Part II) EXAMINATION

MATHEMATICS — [MA412/423]

FOURIER ANALYSIS

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Time allowed: **Two** hours.
Answer *three* questions.

1. (a) Compute the appropriate Fourier transforms of each of the following functions:

(i) $f : \mathbb{R} \to \mathbb{C}$, $f(x) = 1$ on $[-1/2, 1/2]$ and $f(x) = 0$ outside this interval.

(ii) $f : \mathbb{T}_{2\pi} \to \mathbb{C}$, $f(x) = 1$ on $[\pi/2, \pi/2)$ and $f(x) = 0$ on $[-\pi, \pi/2)$ and on $[\pi/2, \pi)$.

(iii) $f : \mathbb{Z} \to \mathbb{C}$, $f[n] = 2^{-n}$ if $n \geq 0$ and $f[n] = 0$ if $n < 0$.

(iv) $f : \mathbb{P}_4 \to \mathbb{C}$, $f = (1, 0, 1, 0)$.

(b) Let f be a function on $\mathbb{P}_N$ with Discrete Fourier Transform F. Prove that

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{2\pi i k n/N} \quad \forall n\,.$$

p.t.o.

2. (a) Let f be an absolutely summable function on $\mathbb{Z}$ (i.e., $\sum_n |f[n]| < \infty$), with Fourier transform $F(s)$. Show that the synthesis formula is valid for f.

(b) Let f be the 2π-periodic function defined by $f(x) = x$ for $-\pi \leq x < \pi$. Show that the Fourier series of f is

$$\sum_{k=1}^{\infty} 2\frac{(-1)^k}{k} \sin kx \, .$$

By estimating the partial sums $S_n(x)$ of this series at the points $\pi - \pi/n$, establish the *Gibbs Phenomenon* for this function.

3. (a) Let f be an integrable function on $\mathbb{R}$ for which the Fourier analysis and synthesis formulas are valid. Suppose f is sampled at intervals of T_0 to generate a function φ on $\mathbb{Z}$, i.e.,

$$\varphi[n] = f(nT_0)\, .$$

Let Φ be the Discrete Time Fourier Transform on ϕ with period $1/T_0$. Show that

$$\Phi(s) = \sum_{m=-\infty}^{\infty} F(s - m/T_0)\, .$$

(You may assume that f is such that this series is uniformly convergent.)

(b) Now suppose that f is *band-limited*, so that the Fourier transform $F(s) = 0$ if $|s| \geq S_0$. Show that if the *Nyquist condition* is satisfied: $1/T_0 > 2S_0$, then f can be recovered from its samples by means of the formula

$$f(t) = \sum_{n=-\infty}^{\infty} \varphi[n] \operatorname{sinc}\left(\frac{t - nT_0}{T_0}\right).$$

(c) Briefly explain what is meant by *aliasing* and give an example to illustrate it.

p.t.o.

4. (a) Give the definition of the convolution product $f * g$ for suitable functions f, g on each of the domains $\mathbb{R}$, $\mathbb{T}_p$, $\mathbb{Z}$ and $\mathbb{P}_N$. For *one* of these domains show that the convolution product corresponds to pointwise (or coordinatewise) mutliplication of the respective Fourier transforms.

   (b) The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1$ on $[-1/2, 1/2]$ and $f(x) = 0$ outside this interval. Compute $f * f$ and hence, or otherwise, find the Fourier transform of the function

$$g(x) = \begin{cases} x+1 & \text{if } \ -1 \le x \le 0\,, \\ 1-x & \text{if } \ 0 \le x \le 1\,, \\ 0 \ \text{ if } \ |x| > 1\,. \end{cases}$$

   (c) Discuss briefly the application of convolution to filtering.

5. (a) Let $N = 2M$ and let f be a function on $\mathbb{P}_N$. Show that the computation of the Discrete Fourier Transform of f can be reduced to the computation of the M-dimensional transforms of the vectors containing the even and odd components, respectively, of f.

   (b) Explain how the above result can be used recursively when N is a power of 2 to yield the Fast Fourier Transform. Illustrate your answer by considering the case $N = 8$.

Fourier Transform Formulas

	Synthesis	Analysis
$f\colon \mathbb{R} \to \mathbb{C}$	$f(x) = \int_{-\infty}^{\infty} F(s) e^{2\pi i s x}\, ds$	$F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x}\, dx$
$f\colon \mathbb{T}_p \to \mathbb{C}$	$f(x) = \sum_{k=-\infty}^{\infty} F[k] e^{2\pi i k x/p}$	$F[k] = \frac{1}{p} \int_0^p f(x) e^{-2\pi i k x/p}\, dx$
$f\colon \mathbb{Z} \to \mathbb{C}$	$f[n] = \int_0^p F(s) e^{2\pi i s n/p}\, ds$	$F(s) = \frac{1}{p} \sum_{n=-\infty}^{\infty} f[n] e^{-2\pi i s n/p}$
$f\colon \mathbb{P}_N \to \mathbb{C}$	$f[n] = \sum_{k=0}^{N-1} F[k] e^{2\pi i k n/N}$	$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-2\pi i k n/N}$