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THE NATIONAL UNIVERSITY OF IRELAND, GALWAY

WINTER EXAMINATIONS 2003

B.Sc. (Part II) EXAMINATION

MATHEMATICS [MA 484]

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Time allowed: **Two** hours.
Full marks for 3 correct solutions.

Q1. (a) Let the random variable X have probability density function $f_X(x|\theta)$, depending on the parameter θ. Let $\widehat{\theta}$ be an estimator of θ based on a random sample $X_1, X_2, \ldots, X_n$ of size n.
Define the following terms:

1. Unbiased estimator
2. Consistent estimator
3. Minimum variance unbiased estimator (MVUE)
4. Mean square error of an estimator
5. Asymptotically unbiased estimator

(b) Let X_1, X_2 be a random sample of an exponential random variable X with parameter λ (> 0), i.e. $f(X|\lambda) = \lambda e^{-\lambda x}, \quad x > 0$. Compare the mean square error of $\overline{X}$ and $U := \sqrt{X_1 X_2}$ as estimators for the mean of X .
Letting $Y := X_1 + X_2$, show that the conditional density of U given $Y = y$ is

$$f_{U|Y=y}(u) = \begin{cases} \frac{4u}{y\sqrt{y^2-4u^2}} & u < y/2 \\ 0 & \text{otherwise} \end{cases}$$

Show that the estimator $\mathrm{E}(U|Y = y)$ has smaller mean square error than U.

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(c) Let $X_1,\ X_2,\ \ldots,X_n$ be a random sample of size n from an exponential distribution with parameter λ as in part (b). Determine whether or not $\sqrt[n]{X_1 X_2 \cdots X_n}$ is a consistent estimator for $\frac{1}{\lambda}$.

Remark: $\frac{d}{dx}\log(\Gamma(x)) = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}\left(\frac{1}{i} - \frac{1}{x+i}\right)$

Q2. (a) State and prove the Cramér–Rao lower bound for the variance of an unbiased estimator $t(X_1, X_2, , \ldots, X_n)$ of a parameter θ, based on a random sample $X_1, X_2, \ldots, X_n$ of size n.

(b) Suppose that t_1 is an unbiased and efficient estimator for θ, and that t_2 is also unbiased, but has efficiency e, satisfying $0 < e < 1$. Show that the correlation coefficient $\rho_{t_1,t_2} = \sqrt{e}$.

(c) Let X_1, X_2 be a random sample of size 2 where X has $\mathrm{N}(0,\ \theta^2)$ density. Compare the relative merits of the two estimators $t_1(X_1,\ X_2),\ t_2(X_1,\ X_2)$ of θ defined as follows:

$$t_1(X_1,\ X_2) := \frac{\sqrt{\pi}}{2}|X_1 - X_2|$$

and

$$t_2(X_1,\ X_2) := \sqrt{\frac{2(X_1^2 + X_2^2)}{\pi}}.$$

Q3. (a) Discuss the method of maximum likelihood as applied to the problem of estimating the parameter of a distribution $f(x|\theta)$. Are maximum likelihood estimators necessarily unique? What are the "optimality" properties enjoyed by maximum likelihood estimators?

(b) Let the random variable X have probability density function

$$f_X(x|\theta_1, \theta_2) := \frac{1}{\theta_1} e^{-\frac{(x-\theta_2)}{\theta_1}} \cdot I_{(\theta_2,\infty)}(x), \quad \theta_1 > 0.$$

Find the method of moments estimators $\widetilde{\theta_1}$ and $\widetilde{\theta_2}$ and also the maximum likelihood estimators $\widehat{\theta_1}, \widehat{\theta_2}$ of the parameters θ_1 and θ_2 based on a random sample $X_1, X_2, \ldots, X_n$ of size n.

(c) Calculate $E(\widehat{X_1}),\ E(\widehat{X_2})$. Does $\widehat{X_2}$ have an asymptotic normal distribution? Comment.

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Q4. (a) Define the following terms:

1. loss function
2. risk function
3. mean risk
4. minimax decision function
5. Bayes decision function with respect to the prior density $\pi(\theta)$ of θ.

(b) **Prove** that a Bayes estimator for a parameter θ having constant risk, is also a minimax estimator.

(c) The random variable X has a $N(\theta,\ 1)$ distribution and the loss function $\mathcal{L}(\theta,\ d)$ is defined as follows:

$$\mathcal{L}(\theta,\ d) = \begin{cases} a(d-\theta) & \text{if} \quad d(X) \geq \theta \\ b(\theta - d) & \text{if} \quad d(X) < \theta \end{cases}$$

where a and b are greater than zero. Calculate the risk of the estimator

$$d_k(X) := X - k.$$

Show that within the class $\mathcal{D} := \{d_k(X) |\ k \in \mathbf{R}\}$, an estimator with uniformly minimum risk exists and that the minimum risk is

$$(a+b)\phi\left\{\Phi^{-1}\left(\frac{a}{a+b}\right)\right\}$$

where ϕ and Φ are respectively the density and cumulative distribution function of the $N(0,\ 1)$ distribution.