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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

EASTER EXAMINATIONS 2004

THIRD UNIVERSITY EXAMINATION

MATHEMATICAL METHODS [MM351] (Repeat Students)

Time allowed: *Three* hours.
Answer *three* questions from each section.
Use separate answer books for each section.

SECTION A

Dr. Dave Johnson,
Professor T. C. Hurley,
Dr. R. A. Ryan

1. (a) Define the terms *Eulerian* and *Hamiltonian* as applied to graphs. Show that, in an Eulerian graph, every vertex has even degree.

   (b) Describe the *Bridges of Königsberg Problem* and show that it has no solution.

   (c) Describe the *Knight's Tour Problem* and show that there is no solution on a $n \times n$ board if n is odd.

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2. (a) Explain the terms *tree*, *spanning tree* and *minimum spanning tree*. The following table shows the distances in miles between Athlone (A), Dublin (D), Galway (G), Limerick (L), Sligo (S) and Wexford (W). Use *Prim's Algorithm* to find a fibre optic network connecting these locations that uses the least amount of cable.

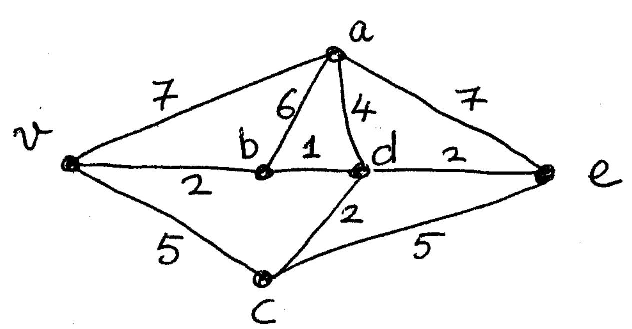
	A	D	G	L	S	W
A		76	56	73	71	114
D	76		132	121	135	96
G	56	132		64	85	154
L	73	121	64		144	116
S	71	135	85	144		185
W	114	96	154	116	185	

(b) Answer **either** (I) **or** (II):

(I) Using a binary decision tree, explain why any algorithm that sorts a list of n numbers must require $O(n \log_2 n)$ operations. (You may use Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$.)
Describe the *Heapsort* algorithm and use it to sort the following list into ascending order:

19, 2, 20, 1, 11, 10, 8, 12, 4.

(II) Apply the *Shortest Path Algorithm* to find the shortest path from vertex v to vertex e in the following graph:



3. (a) Prove *Euler's Formula*, $v - e + f = 2$, for a connected planar graph. Kuratowski's Theorem says that every non-planar graph contains one or other of two particular non-planar graphs. Sketch these two graphs.

(b) Answer **either (I) or (II)**:

(I) Define the *Chromatic Polynomial*, $P_G(k)$, of a graph G, and explain how it can be used to determine the *chromatic number* of G. Find the chromatic polynomial for the graph K_4.

(II) State and prove the *Art Gallery Theorem*. Sketch a gallery for which four guards (and no fewer) are required to watch all the walls.

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4. (i) Sketch the images of the angular sector $0 \leq \theta \leq \pi/4$ (where $z = re^{i\theta}$) under the maps:

(a) $w = iz$; (b) $w = z^2$; (c) $w = iz^2$; (d) $w = 1/z$.

(ii) Find the image of the lower half-plane $y \leq 0$ (where $z = x + iy$) under the Möbius transformation

$$w = \frac{z+i}{z-i}.$$

(iii) Show that the image of a straight line or a circle under a Möbius transformation

$$w = \frac{az+b}{cz+d} \quad \text{(where } ad \neq bc\text{)}$$

is a straight line or a circle.

[You may assume that a straight line or circle is given by an equation of the form $pz\bar{z} + az + \bar{a}\bar{z} + q = 0$, where p and q are real and a is complex.]

5. (i) Give the definition of a *harmonic* function of 2 variables, and show that if $z = x + iy$ and if the function $f(z) = u(x,y) + iv(x,y)$ is differentiable, then $u(x,y)$ and $v(x,y)$ are both harmonic.

[You may assume the Cauchy-Riemann equations.]

Deduce that the inverse tangent function $v(x,y) = \tan^{-1}(y/x)$ is harmonic.

(ii) Find a Möbius transformation which sends -1, i, 1 to 0, i, ∞ respectively.

Hence find a harmonic function $v(x,y)$ defined on the unit disc $x^2 + y^2 < 1$ such that

$$v(x,y) = \begin{cases} \pi/2 & \text{on the upper semi-circle } x^2+y^2=1,\ y>0, \\ -\pi/2 & \text{on the lower semi-circle } x^2+y^2=1,\ y<0. \end{cases}$$

Section B

MM351

Professor B. Straughan;
Dr. M. S. Ó Confhaola;
Professor M. F. McCarthy;
Dr. P. M. O'Leary.

6. Show that the transformation **T** given by

$$\frac{1}{15}\begin{bmatrix} -5 & -14 & 2 \\ -10 & 5 & 10 \\ 10 & -2 & 11 \end{bmatrix}$$

is an orthogonal transformation. A vector field **A** is defined in the x-frame by

$$A_1 = x_1^2, A_2 = x_2^2, A_3 = x_3^2.$$

Evaluate the field in the x'-frame and verify that div**A** is an invariant.

7. Let ε_{ijk} represent the alternating symbol. Evaluate the following determinant:

$$\begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$$

and hence evaluate $\varepsilon_{ijk}\varepsilon_{rst}$.

Show that

(a) $\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$;

(b) $\varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir}$;

(c) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$;

(d) $\varepsilon_{ijk}\varepsilon_{ijm}a_m = 2a_k$.

8. Solve the equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \ : 0 \le x \le 10$$

subject to the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(10,t)}{\partial x} = 0$$

and the initial condition

$$u(x,0) = \begin{cases} 60, & 0 \le x < 5 \\ 0, & 5 \le x \le 10 \end{cases}$$

9. Let $T(r,\theta)$ denote the steady state temperature in the region $1 < r < c$, $0 < \theta < \pi$. The steady state temperature in the region is governed by the equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0, \ \ 1 < r < c, \ \ 0 < \theta < \pi.$$

If $T = T_0$ when $r = c$ and $T = 0$ on all other boundaries, determine the temperature at each point (r,θ) of the region.

10. Find the solution of the one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

in the region $0 \le x \le L$, $t > 0$ which satisfies the boundary conditions

$$y(0,t) = y(L,t) = 0, \ \text{for all } t > 0$$

and the initial conditions

$$y(x,0) = f(x) \ , \ \ \frac{\partial y}{\partial t}(x,0) = g(x).$$

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In the particular case when $g(x) = 0$ show that the solution is equivalent to two progressive waves moving in opposite directions.

11. Solve the equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \quad x > 0, \ t > 0$$

$$u(x,t) \text{ bounded for } x > 0, \ t > 0$$

subject to the initial condition

$$u(x,0) = e^{-x}, x > 0$$

and the boundary condition

$$u(0,t) = 0$$

using the separation of variables and the Fourier integral.
[A function $f(x)$ may be written as

$$f(x) = \int_0^\infty [A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)dx]$$

where

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\bar{x})\cos\alpha\ \bar{x})d\ \bar{x},$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\bar{x})\sin(\alpha\ \bar{x})d\ \bar{x}\ .]$$