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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

FIRST SCIENCE EXAMINATION

MA103 - ALGEBRA

PASS

Second Paper

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Time allowed: **Three** hours.
Answer *six* questions only.

1. Let

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and } B = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}.$$

(a) Find AB.
(b) Find A^{-1} and B^{-1}.
(c) Find a 2×2 matrix C so that $AC = B$.
(d) Show that $|AB| = |A||B|$ (where $|A|$ is the determinant of A).

p.t.o.

2. For the transformation of the plane defined by the matrix $\begin{pmatrix} -3 & 4 \\ 2 & 1 \end{pmatrix}$, find

(a) the image of the point $(1, -1)$,

(b) the image of the line $x + y = 3$,

(c) the line whose image is $2x + y = 2$.

3. Let $A = \begin{pmatrix} -3 & 4 \\ 2 & -1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find a diagonal matrix D and an invertible matrix E such that $AE = ED$.

(c) Calculate A^{100}.

4. (a) Reduce the conic $4x^2 + 16x + 9y^2 - 18y - 11 = 0$ to standard form and sketch its graph.

(b) Prove by induction that $-2 + 2(-5)^n$ is divisible by 3.

p.t.o.

5. (a) Indicate on Argand diagrams the set of points which satisfy:

(i) $z = \bar{z}$

(ii) $-\frac{\pi}{4} \leq Arg(z) \leq \frac{\pi}{3}$

(iii) $|z - i| = |z - 3i|$

(b) Calculate

$$\left(\frac{\sqrt{2} - \sqrt{2}i}{1+i}\right)^{10}$$

expressing your answer in the form $a + bi$.

(c) Verify that $z = 1 + i$ is a root of

$$f(z) = z^3 + (-1 - 2i)z^2 + (3i - 4)z + (i + 5).$$

Verify that $f(z) = (z - (1 + i))(z^2 - iz + 2i - 3)$.
Is $z = i$ a root of $f(z)$?

6. (a) Find, using De Moivre's Theorem or otherwise, the eight roots of unity $(\omega_0, \omega_1, ..., \omega_7)$.

(i) Hence express $x^8 - 1$ as a product of linear (complex) factors, and

(ii) Plot these eight roots on an Argand diagram.

(b) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z - z^{-1} = 2i\sin\theta$.

(ii) By expanding $(z - z^{-1})^3$ and using De Moivre's Theorem, show that

$$-8i\sin^3\theta = 2i\sin 3\theta - 4i\sin\theta.$$

p.t.o.

7. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

(a) Find the matrix of cofactors of A.

(b) Find the adjoint A^* of A.

(c) Compute the product AA^* and state the determinant of A.

(d) Find A^{-1}.

(e) Use (d) to solve

$$\begin{aligned} x \quad + 2z &= 2 \\ 3x + y - z &= 1 \\ 2x + y + z &= -1 \end{aligned}$$

8. Suppose that a city has a fixed population but in each year, $\frac{1}{2}$ of the people in the city centre move out to the suburbs and $\frac{1}{4}$ of the people in the suburbs move into the city centre. Let x_n and y_n represent the fraction of the city's population resident in the city centre and in the suburbs respectively, after n years (thus $x_n + y_n = 1$).

(a) Find the transition matrix T for this process.

(a) Show that 1 and $\frac{1}{4}$ are eigenvalues of T and find the corresponding eigenvectors for each eigenvalue.

(c) Explain the term "steady state" and find the steady state in this problem.

(d) Show that x_n and y_n tend to the steady state values as $n \to \infty$, regardless of the values of x_0 and y_0.