GO2003

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004

FIRST ENGINEERING & INFORMATION TECHNOLOGY EXAMINATION

MATHEMATICS [MA150]

MA153 — *ALGEBRA*

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Time allowed: *Three* hours.
Answer six questions.

1. (a) Find the general solution of the following system of equations.

$$\begin{array}{rcrcrcrcr} x & + & 2y & + & 2z & + & w & = & -1 \\ x & + & 2y & + & z & & & = & -1 \\ -2x & + & -4y & & & + & 3w & = & 3 \end{array}$$

(b) Consider the following system of equations in variables x, y and z.

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 1 \\ 2x & & & + & 3z & = & 1 \\ & - & 4y & + & kz & = & s \end{array}$$

(i) If the system has no solutions, what is the value of k?
(ii) If the system has infinitely many solutions, what is the value of s?

p.t.o.

2. (a) Calculate the adjoint of the matrix

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

and hence, or otherwise, determine whether or not A^{-1} exists.

(b) Write down the determinant of the following matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

3.

(a) Evaluate $\begin{vmatrix} 1 & -4 & 4 & 0 \\ -5 & 3 & 4 & -2 \\ 1 & -1 & 3 & 4 \\ -1 & 1 & -3 & 3 \end{vmatrix}.$

(b) Show that $\begin{vmatrix} a & b & c \\ b & a & b \\ c & c & a \end{vmatrix} = (a+b+c)(b-a)(c-a)$. Hence or otherwise find $\begin{vmatrix} 3 & 5 & -4 \\ 5 & 3 & 5 \\ -4 & -4 & 3 \end{vmatrix}.$

4. (a) Let $A = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 3 & -1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ -3 & 2 & 1 & 0 \end{pmatrix}$. Compute the matrix products AB and BA, if defined.

(b) Use the Gauss-Jordan method to calculate the inverse of B. Check your answer by verifying that $B^{-1}B = I_4$.

(c) Suppose that A and B are orthogonal $n \times n$ matrices. Show that AB is orthogonal.

p.t.o.

5.

(a) Sketch Argand diagrams to represent the complex numbers z which satisfy each of the following conditions:
   (i) $z+i=\bar{z}-i$.
   (ii) $-\frac{\pi}{3} \leq \mathrm{Arg}(z) \leq \frac{\pi}{3}$.
   (iii) $1 < |z-2| < 3$.

(b) Prove the triangle inequality: $|z+w| \leq |z|+|w|$, for all $z, w \in \mathbb{C}$.

(c) Use de Moivre's Theorem and the Binomial Theorem to find integers a, b, c such that

$$\cos 4\theta = a\cos^4\theta + b\cos^2\theta + c.$$

(d) Express $(1-i)^{35}$ in the form $x+iy$.

6.

(a) Find all the roots of $z^4-12z^3+57z^2-120z+100$, given that one root is twice another root.

(b) Write down the 6^{th} roots of unity in the form $a+bi$. Illustrate these roots on the argand plane. Hence write z^6-1 as the product of real linear and real irreducible quadratic factors.

(c) Explain why z^2-1 is a factor of z^n-1 if and only if n is even. When will z^2+1 be a factor of z^n+1?

7.

(a) The matrix M has an eigenvalue of $\lambda = 3$ with associated eigenvector $v = (2,-1,0,3)^T$. Write down the vector Mv.

(b) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 4 & -2 \\ -3 & 5 & 0 \\ -3 & 1 & 4 \end{pmatrix}.$$

Write down a diagonal matrix D and a matrix E such that $E^{-1}AE = D$.

(c) Let B be a matrix with eigenvalue λ and let k be a constant. Show that the matrix $C = B + kI$ has eigenvalue $\lambda + k$, if I is the identity matrix of the same size as B.

p.t.o.

8.

(a) Sketch the curve given by the equation

$$4x^2 - y^2 + 8x - 4y = 8.$$

(b) Find a standard form equation for the ellipse with foci $(\pm 4, 0)$ and directrix $x = \frac{16}{3}$.

(c) Find an orthogonal transformation which reduces the conic

$$x^2 - 3xy + y^2 = 5$$

to standard form.