GD2004

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004

FIRST UNIVERSITY EXAMINATION

MATHEMATICS [MA180]
MA181 - Analysis

HONOURS
First Paper

Dr. Dave Johnson
Prof. T. Hurley
Prof. M. Newell
Dr. J. McDermott

Time allowed: *Three* hours.
Full marks for six questions.

Q1. (a) Explain why the function $f(x) = \frac{\sqrt{x^2-1}}{x+1}$ is not defined when $-1 \leq x < 1$.

(b) Show $f(x)$ is negative when $x < -1$.

(c) Sketch the curve $y = f(x)$ and include its horizontal and vertical asymptotes.

Q2. (a) Using the definition $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$,
prove $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$, when $g'(a) \neq 0$ and $f(a) = g(a) = 0$.

(b) Evaluate the following limits:

(i) $\lim_{x \to 1} \frac{x^2-1}{2-\sqrt{x^2+3}}$ (ii) $\lim_{\theta \to 0} \frac{1-\cos 2\theta}{\sin\theta \sin 2\theta}$ (iii) $\lim_{x \to \infty} \frac{x^2-x+1}{2x^2-9}$.

pto

Q3. (a) Find the derivative y' for the functions defined by:

(i) $y = \tan^{-1}(\sqrt{x+1})$ (ii) $y = \sqrt{\dfrac{x-1}{x+1}}$

(iii) $y\,(\sin x - y) = \cos x$.

(b) A right cylindrical closed can has base radius r and height h. Express the volume V and surface area S in terms of r and h. Prove that when the volume V_0 is fixed, the can with least surface area has height equal to its base diameter.

Q4. (a) Evaluate three of the following integrals:

(i) $\int \frac{x-2}{\sqrt{x^2-4x}}\,dx$ (ii) $\int (\tan^{-1}x)x\,dx$ (iii) $\int \frac{dx}{1+\cos 2x}$ (iv) $\int \sec^3 x\,dx$.

(b) Let $I_n = \int \cos^n x\,dx$, where n is a positive integer. Find I_1 and prove the reduction formula

$$\frac{n}{n-1} I_n = I_{n-2} + \frac{1}{n-1}(\cos^{n-1} x \sin x) \text{ when } n > 1.$$

Hence evaluate $\int \cos^3 dx$.

Q5. (a) (i) Find $\lim\limits_{n\to\infty} \frac{2n-1}{n+1}$.

(ii) Give the definition for $\lim\limits_{n\to\infty} a_n = L$, and show that it is satisfied in (i) above.

(b) State the Completeness Axiom for $\mathbb{R}$, and explain the terms involved. Find (with proof) the supremum $\sup S$ ($= \text{lub}\, S$) of the set $S = \{2 - \frac{3}{n} : n \epsilon \mathbb{N}\}$.

Q6. (a) State the main limit theorems and use them to show that

(i) $\lim\limits_{x\to 2} \frac{3x+4}{x^2+x-1} = 2$ (ii) $\lim\limits_{x\to 1} \frac{x^2-4}{x-1}$ does not exist.

(b) Show that if the function f is differentiable at c then f is continuous at c. Show that the function g defined by $g(x) = |x^2 - 1|$ is continuous at 1 but not differentiable at 1.

pto

Q7. (a) State the Intermediate Value Theorem (IVT) and the Mean Value Theorem (MVT). Use IVT and/or MVT (as appropriate) to show that:

(i) if $f'(x) > 0$ for all x in (p, q) then the function f is increasing on (p, q);

(ii) the equation $x = 2 + \ln x$ has one, and only one, solution in $[1, 4]$.

(b) Sketch a proof of IVT.

Q8. (a) The function h is defined by $h(x) = (1+x^2)^{-1}$, and P is the partition of $[0, 1]$ into eight equal parts. Calculate the lower Riemann sum $L(h, P)$ and hence show that $\pi > 3$.

(b) (i) Show that the improper integral $\int_1^\infty \frac{dx}{(x+2)^2}$ converges (exists), and find its value.

(ii) Show that

$$\frac{1}{4^2} + \frac{1}{5^2} + \dots + \frac{1}{(n+2)^2} \leq \int_1^n \frac{dx}{(x+2)^2},$$

and deduce that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+2)^2}$ is convergent.

Estimate the sum L of this series in the form $a \leq L \leq b$.