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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER TWO EXAMINATIONS, 2003 - 2004

SECOND ENGINEERING EXAMINATION

MATHEMATICS (MA250 = MA253/254)

Dr. Dave Johnson,
Prof. T.C. Hurley,
Paul Wilson, M.A., H.D.E.

Time allowed: *Three hours*

Attempt *five* questions, at least two from each section.
It is **not** necessary to use separate answer books for each section.

SECTION A

1. (a) i. Show that $\cos iz = \cosh z$ and $\sin iz = i \sinh z$.

   ii. hence express $\cosh z$ in the form $u(x,y) + iv(x,y)$ and verify that $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann conditions.

   (b) Show that the function

$$f(z) = \overline{z}$$

   is *not* differentiable.

   (c) Let $v(x,y)$ be a harmonic conjugate of the function $u(x,y)$ *and* $u(x,y)$ be a harmonic conjugate of $v(x,y)$. Show that both $u(x,y)$ and $v(x,y)$ are constant functions.

Questions 2 and 3 are on the next page

2. (a) Evaluate the line integral:

$$\int_{\Theta} z^2 dz$$

where Θ is:

i. the line segment from $z = 0$ to $z = 2 + i$,

ii. the upper half of the circle $|z| = 2$.

(b) Let C be the path $z = e^{it}$, $0 \leq t \leq 2\pi$. Show directly, (i.e. without using the method of residues or Cauchy's integral formula) that:

$$\int_C z^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

3. (a) Assume Cauchy's formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0}$$

where C is a simple closed contour within which $f(z)$ is differentiable, and z_0 is an interior point of C. Outline a proof that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{n+1}}$$

stating any assumptions that you make.

(b) Hence evaluate the following integrals:

$$\text{(i)} \int_C \frac{zdz}{(2 + z)(z - i)} \qquad \text{(ii)} \int_C \frac{z^2 - i \sin z}{(z - \frac{\pi}{4}i)^5} dz$$

where C is the contour $|z| = \frac{3}{2}$.

Questions 4 and 5 are on the next page

4. (a) i. Find Laurent series for the function

$$h(z) = \frac{1}{z(z+2)^3}$$

that converge for:

A. $0 < |z| < 2$,

B. $0 < |z+2| < 2$

ii. Using the Laurent series obtained, explain why $h(z)$ has a simple pole at $z = 0$ and a pole of order three at $z = -2$.

iii. Hence, or otherwise, evaluate

$$\int_\Gamma h(z)dz$$

where Γ is the positively oriented contour defined by the boundary of the rectangle with vertices at $4 \pm i$ and $-4 \pm i$, and $h(z)$ is as above.

(b) Recall that the *Taylor Series*, (about $\theta = 0$), of the function $\cos\theta$ is $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$

Find the Laurent series for the function

$$f(z) = z^2 \cos\left(\frac{1}{z}\right)$$

about the singularity $z = 0$. Is this singularity a removable singularity, a pole or an essential singularity? Justify your answer.

5. Use the method of residues to evaluate any *two* of the following integrals:

(a)

$$\text{(i)} \int_0^{2\pi} \frac{d\vartheta}{2 - \sin\vartheta} \qquad \text{(ii)} \int_0^\infty \frac{\cos x}{x^2+1}dx \qquad \text{(iii)} \int_{-\infty}^\infty \frac{x^2}{x^4+1}dx$$

(b) Explain why we may *not* use the method of residues to evaluate:

$$\int_0^\pi \frac{d\vartheta}{2 - \sin\vartheta}$$

Questions 6,7 and 8 are on the next page

SECTION B

6. (a) Explain what is meant by the *orthogonal complement*, $W^{\perp}$, of a subspace W of R^n.

   (b) Find a basis in R^4 for the orthogonal complement of the subspace

$$W = sp\{(1,3,4,5),(1,3,6,7)\}$$

   (c) Show that any vector $\vec{b} \in R^n$ may be expressed in the form

$$\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$$

   (d) i. Find the projection, $\vec{b}_W$, of $\vec{b} = (2,1,-4)$ onto $W = sp\{(1,2,1),(2,1,-1)\}$
   ii. What is $\vec{b}_{W^\perp}$ in this case? Verify that $\vec{b} = \vec{b}_W + \vec{b}_{W^\perp}$.

7. (a) Write down the projection matrix that projects vectors in R^3 onto the plane spanned by the vectors $\vec{v}_1 = (1,1,1)$, $\vec{v}_2 = (1,2,3)$, and hence find the projection of the vector $\vec{w} = (1,0,1)$ onto this plane.

   (b) Explain what is meant by a basis $\left\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \ldots\ldots, \vec{b}_n\right\}$ of a subspace W of R^n being *orthonormal.*

   (c) Use the Gram–Schmidt process to find an orthonormal basis $\{\vec{u}_1, \vec{u}_2\}$ for the plane spanned by the vectors $\vec{v}_1$ and $\vec{v}_2$ from part (a) above, and hence confirm your result concerning the projection of the vector $\vec{w} = (1,0,-1)$ onto this plane.

8. (a) Let M be a real symmetric matrix. Prove that the eigenvectors corresponding to distinct eigenvalues of M are orthogonal.

   (b) Let

$$G = \begin{pmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

   i. Show that the characteristic equation of G is $(\lambda - 1)^2(\lambda - 10)$.
   ii. Determine the eigenvalues and corresponding eigenvectors of G.
   iii. Write down an orthogonal matrix Q and a diagonal matrix D such that $Q^TGQ = D$. [You need only write down the matrices Q and D, there is no need to verify that $Q^TGQ = D$].

Question 9 is on the next page

9. $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$ defines an inner product on $P[-1,1]$, the set of all polynomials on $[-1,1]$.

(a) Commencing from the basis $\{1, x, x^2\}$ for $P_2[x]$, the space of all polynomials of degree two on $[-1,1]$, (i.e. the space of all functions of the form $ax^2 + bx + c$, where $-1 \le x \le 1$), use the Gram–Schmidt process to derive the ortho*gonal* basis $\left\{1, x, x^2 - \frac{1}{3}\right\}$ for this space, with respect to the above inner product.

(b) Hence find an ortho*normal* basis for the above space.

(c) Find the projection of the function $f(x) = x^4$ onto $P_2[x]$ on $[-1,+1]$. Interpret your answer.