

*Ollscoil na hÉireann, Galway*  
*National University of Ireland, Galway*

GX 2014

**Spring Examinations, 2003/2004**

Exam Code(s)	3CS1,4IF1
Exam(s)	Third Science and Fourth Information Technology
Module Code(s)	CS305II
Module(s)	Computing Techniques Of Applied Mathematics
Paper No	1
Repeat Paper	Special Paper
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Dr. M.G. Meere
Instructions:	Attempt <b>THREE</b> questions
Duration	<b>TWO HOURS</b>
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3 (excluding this page)
Department(s)	Mathematical Physics

1. Consider the following two point boundary value problem for a function  $u = u(x)$ :

$$-\frac{d^2u}{dx^2} - u + x^2 = 0 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 1 \text{ on } x = 1.$$

- a. Show that the weak form of this boundary value problem is given by

$$B(u, w) = l(w)$$

where

$$B(u, w) = \int_0^1 \left\{ \frac{du}{dx} \frac{dw}{dx} - uw \right\} dx \text{ and } l(w) = -\int_0^1 x^2 w dx + w(1)$$

and where  $w = w(x)$  is a sufficiently smooth weight function. In your derivation you should identify the primary variable and the secondary variable, as well as classifying the boundary conditions.

- b. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_N\phi_N(x)$$

where the  $\{\phi_i(x)\}_{i=1}^N$  are pre-selected approximation functions, show that for the Rayleigh-Ritz procedure, the  $\{c_i\}_{i=1}^N$  are determined by solving the linear equations

$$\sum_{j=1}^N B_{ij}c_j = F_i \text{ for } i = 1, 2, \dots, N$$

where

$$B_{ij} = B(\phi_i, \phi_j), F_i = l(\phi_i) - B(\phi_i, \phi_0).$$

- c. Calculate the Rayleigh-Ritz approximation to the above problem for the case  $N = 2$  with

$$\phi_0(x) = 0, \phi_1(x) = x, \phi_2(x) = x^2.$$

2. Consider the following two point boundary value problem for a function  $u = u(x)$ :

$$A(u) = f(x), 0 < x < 1,$$

$$\frac{du}{dx} = 0 \text{ on } x = 0 \text{ and } u = 0 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + 16u, f(x) = 4.$$

- a. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_N\phi_N(x)$$

where the  $\{\phi_i(x)\}_{i=1}^N$  are pre-selected approximation functions, show that for the Galerkin method, the  $\{c_i\}_{i=1}^N$  are determined by solving the linear equations

$$\sum_{j=1}^N A_{ij}c_j = F_i \text{ for } i = 1, 2, \dots, N$$

where

$$A_{ij} = \int_0^1 \phi_i A(\phi_j) dx, F_i = \int_0^1 \phi_i (f - A(\phi_0)) dx.$$

- b. Calculate the Galerkin approximation to the above problem for the case  $N = 2$  with  $\phi_0(x) = 0, \phi_1(x) = \cos(\pi x/2), \phi_2(x) = \cos(3\pi x/2).$

3. Consider the following two point boundary value problem for a function  $u = u(x)$ :

$$A(u) = f(x), 0 < x < 1,$$

$$u = 1 \text{ on } x = 0 \text{ and } \frac{du}{dx} = -1 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + xu, f(x) = 1 - x.$$

- a. An approximate solution to this problem is sought of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_N\phi_N(x)$$

where the  $\{\phi_i(x)\}_{i=1}^N$  are pre-selected approximation functions. The residue,  $R$ , is defined by  $R = A(U_N) - f$ . Show that by minimizing  $\int_0^1 R^2 dx$  with respect to the  $c_j$ 's,  $j = 1, 2, \dots, N$ , that (the least-squared method)

$$\sum_{j=1}^N A_{ij}c_j = F_i \text{ for } i = 1, 2, \dots, N$$

where

$$A_{ij} = \int_0^1 A(\phi_i)A(\phi_j)dx, F_i = \int_0^1 A(\phi_i)(f - A(\phi_0))dx.$$

- b. Calculate the least-squared approximation to the above problem for the case  $N = 1$  with

$$\phi_0(x) = 1 - x, \phi_1(x) = x(x - 2).$$

4. Consider the following two point boundary value problem for a function  $u(x)$ :

$$-\frac{d^2u}{dx^2} + u = 2 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 1 \text{ on } x = 1.$$

The interval  $0 < x < 1$  is split into three equal subintervals of length  $1/3$  with a view to estimating the solution using the Finite Element Method.

The elements are denoted by  $(x_{1,e}, x_{2,e})$  with  $e = 1, 2, 3$ . Estimating  $u$  in the interval  $(x_{1,e}, x_{2,e})$  by the linear approximation

$$U_e = u_{1,e}\psi_{1,e}(x) + u_{2,e}\psi_{2,e}(x)$$

where  $\psi_{1,e}(x) = 3(x_{2,e} - x)$ ,  $\psi_{2,e}(x) = 3(x - x_{1,e})$ , the element equations are given by (you are not required to show this):

$$\sum_{j=1}^2 K_{ij,e}u_{j,e} - f_{i,e} - (-1)^i Q_{i,e} = 0 \text{ for } i = 1, 2 \text{ and } e = 1, 2, 3,$$

where

$$K_{ij,e} = \int_{x_{1,e}}^{x_{2,e}} \left( \frac{d\psi_{i,e}}{dx} \frac{d\psi_{j,e}}{dx} + \psi_{i,e}\psi_{j,e} \right) dx, f_{i,e} = \int_{x_{1,e}}^{x_{2,e}} 2\psi_{i,e} dx \text{ for } i, j = 1, 2 \text{ and } e = 1, 2, 3,$$

$$Q_{1,e} = \left( \frac{du}{dx} \right)_{x=x_{1,e}}, Q_{2,e} = \left( \frac{du}{dx} \right)_{x=x_{2,e}} \text{ for } e = 1, 2, 3.$$

- a. Show that

$$\mathbf{k}^e = (K_{ij,e}) = \begin{pmatrix} 28/9 & -53/18 \\ -53/18 & 28/9 \end{pmatrix}, \mathbf{f}^e = (f_{i,e}) = \begin{pmatrix} 1/3 \\ 1/3 \end{pmatrix} \text{ for } e = 1, 2, 3.$$

- b. Writing  $u_{2,1} = u_{1,2} = U_2$ ,  $u_{2,2} = u_{1,3} = U_3$ ,  $u_{2,3} = U_4$ , show that

Cont'd overleaf.

Q4 continued.

$$\begin{pmatrix} 56/9 & -53/18 & 0 \\ -53/18 & 56/9 & -53/18 \\ 0 & -53/18 & 28/9 \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 2/3 \\ 4/3 \end{pmatrix}.$$

5. a. Consider the following system of linear equations:

$$\begin{pmatrix} 1 & 4 & 4 \\ 0 & 2 & 3 \\ 1 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -8 \end{pmatrix}$$

- i. Write a MAPLE command to define the coefficient matrix A (say) in the above system of equations.
  - ii. Write MAPLE commands to calculate  $A^{-1}$ ,  $A^t$ ,  $A^{15}$  and the determinant of A.
  - iii. Write MAPLE commands to calculate the eigenvalues and eigenvectors of A.
  - iv. Write MAPLE commands to calculate the characteristic polynomial of A, and to verify that A satisfies this characteristic polynomial.
  - v. Write MAPLE commands to solve the linear system of equations defined above.
- b. Write MAPLE commands to solve each of the following problems:

i.  $\frac{dy}{dt} = y^5$ ,  $y(0) = 1$ ,

ii.  $-\frac{d^4y}{dx^4} + y = x$ ,  $y(0) = 1$ ,  $\frac{dy}{dx}(0) = 0$ ,  $y(1) = 0$ ,  $\frac{dy}{dx}(1) = 1$ .