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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

FIRST ARTS EXAMINATION

MA123 - ALGEBRA

Dr D. Johnson
Prof T. Hurley
Dr P. Kirwan
Dr G. Ellis

Time allowed: *three* hours.

Answer *five* questions.

1. Let $A = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & -2 \\ 1 & 3 \end{pmatrix}$.

   (a) (i) Calculate the inverses A^{-1} and B^{-1}.

   (ii) Verify that $(BA)^{-1} = A^{-1}B^{-1}$

   (b) (i) Use A^{-1} to solve the system of equations

$$\begin{aligned} 3x - 2y &= 7 \\ 2x - y &= 3 \end{aligned}$$

   (ii) Use B^{-1} to solve the matrix equation $XB = A$

   (c) Determine the image of the unit square under a rotation of $\frac{\pi}{3}$ radians. Sketch the image of the unit square.

P.T.O.

2. Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

(a) Calculate A^2 and $|A|$, and determine t such that $A^2 + |A|I = tA$

(b) (i) Determine the eigenvalues and eigenvectors of A.

(ii) Hence, write down a diagonal matrix D and a non-singular (invertible) matrix E such that $AE = ED$.

(iii) Hence, or otherwise, calculate A^6.

3. (a) Prove by induction that

$$n! > 2^n \text{ for } n \geq 4$$

(Note: $n! = n(n-1)(n-2)\cdots(2)(1)$)

(b) (i) Use the Euclidean Algorithm to calculate $gcd(204, 171)$ and express it in the form $204s + 171t$ with s and t in $\mathbf{Z}$.

(ii) Determine integers x and y such that $204x + 171y = 12$.

(iii) Show that there are no integers satisfying $204x + 171y = 10$.

4. A school of 1000 students is quarantined due to the presence of a contagious disease. Each day 20% of those that are well become ill and 30% of those that are ill get well.

(a) If x_n and y_n are the proportions that are ill and well, respectively, after n days (with x_0 and y_0 the initial proportions) then show that $A = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$ is the transition matrix for the process such that

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = A \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

(b) Determine the eigenvalues and eigenvectors of the transition matrix A and calculate A^n.

(c) Determine the steady-state vector of A and hence calculate the number of students that will be ill in the long-term.

(d) If nobody is ill initially, then what proportion of the student population will be well after 2 days?

5. (a) Calculate $19^{-1} \bmod 26$.

   (b) Calculate $A^{-1} \bmod 26$ where $A = \begin{pmatrix} 14 & 11 \\ 17 & 10 \end{pmatrix}$. Express each of the four entries in A^{-1} as an integer from 0 to 25.

   (c) The ciphertext

$$NAYQHLQV$$

   is written in a 26-letter alphabet (A=0, $\cdots$,Z=25). By applying the deciphering function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 14 & 11 \\ 17 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

   to pairs of letters, determine the *first four* letters of plaintext.

6. (a) Calculate A^{-1} where $A = \begin{pmatrix} 2 & 2 & 7 \\ 1 & 1 & 3 \\ 2 & 1 & 6 \end{pmatrix}$.

   (b) Solve the following system of linear equations.

$$\begin{array}{ccccccc} 2x & + & 2y & + & 7z & = & 27 \\ x & + & y & + & 3z & = & 12 \\ 2x & + & y & + & 6z & = & 22 \end{array}$$

   (c) Use the equality $(AB)^t = B^t A^t$ to prove that $(A^t)^{-1} = (A^{-1})^t$. Then solve the following system of linear equations.

$$\begin{array}{ccccccc} 2x & + & y & + & 2z & = & 5 \\ 2x & + & y & + & z & = & 4 \\ 7x & + & 3y & + & 6z & = & 16 \end{array}$$

7. (a) Find the modulus and argument of the complex number

$$w = \frac{1 + \sqrt{3}\mathrm{i}}{1 - \sqrt{3}\mathrm{i}}$$

   and then express w^{10} in the form $r(\cos\theta + \mathrm{i}\sin\theta)$.

   (b) Factorize $x^5 + x^4 + x^3 + x^2 + x + 1$ as a product of real linear and quadratic factors.

   (c) Deduce that

$$\sin\frac{\pi}{3} + \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} + \sin\frac{5\pi}{3} = 0.$$

   from the fact that the sixth roots of unity sum to zero.