GX2032

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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

SECOND UNIVERSITY EXAMINATION

MATHEMATICS [MA 283]

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Time allowed: Two hours.
Full marks for 3 correct solutions.

Q1. (a) Let $\mathbf{V}$ be a finite dimensional vector space over the real numbers $\mathbf{R}$. Define the terms (i) *spanning set* S, (ii) *linearly independent set* I and (iii) *basis* B where S, I and B are finite subsets of $\mathbf{V}$.

(b) Consider the 5×5 matrix A below:

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 1 & 3 & 1 & 1 & -1 \\ 2 & 5 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & -6 \\ 1 & 5 & 3 & 5 & -5 \end{pmatrix}$$

Determine the *column rank* of A. Find also a basis for the *null space* of A. Explain why the *row rank* of A is equal to its column rank.

(c) Let A be an $m \times n$ matrix of *rank* r, which is of course less than or equal to $\min\{m, n\}$. Prove that $A^{\mathrm{T}}A$ has the same rank as A.

Q2. (a) Let f be a linear mapping from the finite dimensional vector space $\mathbf{V}$ into the finite dimensional vector space $\mathbf{W}$. Define *ker*(f), the *kernel* and *Im*(f), the *image* of the linear map f.
Prove that *ker*(f) is a subspace of $\mathbf{V}$ and that *Im*(f) is a subspace of $\mathbf{W}$.

(b) Let $f : \mathbf{R}^5 \mapsto \mathbf{R}^3$ be defined by $f(a,b,c,d,e) = (-3a+6b-c+d-7e, a-2b+2c+3d-e, 2a-4b+5c+8d-4e)$. Write down the 3×5 matrix corresponding to this mapping f and hence or otherwise find a basis for *Im* (f) and *ker*(f).

(c) Expand the basis of *Im*(f) in part (b) to obtain a basis for $\mathbf{R}^3$.

Q3. (a) Let f be a linear mapping from a vector space $\mathbf{V}$ which has ordered basis $\{v_1, v_2, \dots, v_m\}$ to a vector space $\mathbf{W}$ which has ordered basis $\{w_1, w_2, \dots, w_n\}$. Let f have the corresponding $n \times m$ matrix A.
Suppose $\{v_1', v_2', \dots, v_m'\}$ and $\{w_1', w_2', \dots, w_n'\}$ are new ordered bases for $\mathbf{V}$ and $\mathbf{W}$ respectively, with corresponding transition matrices P from $\{v_1', v_2', \dots, v_m'\}$ to $\{v_1, v_2, \dots, v_m\}$ and Q from $\{w_1', w_2', \dots, w_n'\}$ to $\{w_1, w_2, \dots, w_n\}$ respectively. Explain how to find the matrix of f relative to the new ordered bases $\{v_1', v_2', \dots, v_m'\}$ and $\{w_1', w_2', \dots, w_n'\}$.

(b) Let $f : \mathbf{R}^3 \mapsto \mathbf{R}^2$ be defined by $f(a,b,c) = (a+c, b-c)$. Write down the matrix A of f relative to the natural ordered basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbf{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbf{R}^2$.
Relative to the new ordered basis $\{(1,1,0),(0,1,1),(0,0,1)\}$ of $\mathbf{R}^3$ and $\{(1,1),(1,3)\}$ of $\mathbf{R}^2$ respectively, find the matrix B of f and verify the result in part (a).

Q4. (a) Prove that if A is an $n \times n$ matrix over $\mathbf{R}$ with n distinct real eigenvalues then A is diagonalizable (ie there exists an invertible matrix E such that $E^{-1}AE$ is diagonal).

(b) For the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

find a matrix E such that

$$E^{-1}AE = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Let A_n be the $n \times n$ matrix given by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

Prove that $\det A_n = (-1)^{n-1}(n-1)$.

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Q5. (a) Describe the **Gram–Schmidt** process for converting a basis for a finite dimensional Euclidean vector space $\mathbf{V}$ into an *orthonormal* basis for $\mathbf{V}$.

(b) Let S be a real symmetric matrix. Prove that the eigenvectors corresponding to distinct eigenvalues of S are orthogonal.

(c) Let

$$S = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Find an orthogonal matrix O such that

$$O^{\mathrm{T}} S O = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$