GX2033

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

SECOND ARTS and SCIENCE EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS [MA287 — COMPLEX ANALYSIS]

HONOURS

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Time allowed: *Two* hours.
Answer *three* questions

1. (a) Give the definition of the exponential function e^z and use it to prove Euler's formula. Sketch the image under the mapping $z \mapsto e^z$ of the region

$$\{z \in \mathbb{C} : \pi/4 \leq \operatorname{Im} z \leq \pi/2\}.$$

(b) Define the principal branch of the complex logarithm function $\operatorname{Log} z$, and show that $e^{Log z} = z$. Hence find the derivative of $\operatorname{Log} z$.

(c) Given that the stereographic projection z of the point (θ, ϕ) (in spherical polar coordinates) on the Riemann sphere is given by $z = cot(\phi/2)e^{i\theta}$, prove that the mapping $f(z) = 1/z$ on $\mathbb{C}$ induces a rotation of the Riemann sphere about the real axis, through an angle π.

2. (a) Let $f = u + iv$ be holomorphic on an open subset Ω of $\mathbb{C}$. Prove that ∇u and ∇v are of equal length and are perpendicular.

(b) Let $h(u, v)$ be a harmonic function of u and v on $f(\Omega)$. Prove that $g(x, y) = h(u(x, y), v(x, y))$ defines a harmonic function on Ω.

(c) Find a Möbius transformation that maps the points $z_1 = 0, z_2 = 1, z_3 = \infty$, onto the points $w_1 = -1, w_2 = -i, w_3 = 1$.

p.t.o.

3. (a) State the *Fundamental Theorem* for the integral $\int_\gamma f(z)\,dz$, where γ is a smooth path in the complex plane. Evaluate the integral

$$\int_\gamma \frac{dz}{(z-a)^n},$$

where n is a natural number and γ is the circle with centre a and radius r, traversed anticlockwise. You should distinguish between the cases $n = 1$ and $n > 1$.

(b) State and prove *Cauchy's Theorem* for homotopic paths.

4. (a) State the *Cauchy Integral Formula*, and give a heuristic proof of it. Evaluate the integral

$$\int_\gamma \frac{1}{(z-2i)^3(z-i/2)}\,dz,$$

where γ is the circle $|z| = 1$.

(b) State the *Residue Theorem*. Find the residues of the function

$$f(z) = \frac{z}{(z^2+4z+1)^2}$$

at each of its poles. Hence evaluate the integral

$$\int_0^{2\pi} \frac{1}{(2+cos\theta)^2}\,d\theta.$$