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SEMESTER II EXAMINATIONS 2004

MATHEMATICS [MA344] — GROUPS II

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Time allowed: *Two* hours.
Answer *three* questions.

1. Let $\mathcal{A} = (X, A, \delta, x_0, F)$ be an automaton with state set X, input alphabet A, initial state x_0, accepting states F and transition function δ.

(a) Draw a diagram of the automaton $\mathcal{A}$ with $X = \{0, 1, 2, 3\}$, $A = \{a, b\}$, $x_0 = 0$, $F = \{2, 3\}$ and δ given by the table:

x	$\delta(x, a)$	$\delta(x, b)$
0	1	2
1	0	2
2	1	2
3	0	2

(b) What is a monoid action? Show that the map $\delta^* \colon X \times A^* \to X$ defined inductively as $\delta^*(x, \epsilon) = x$ for the empty word ϵ and $\delta^*(x, wc) = \delta(\delta^*(x, w), c)$ for $x \in X$, $c \in A$ and $w \in A^*$ is an action of the free monoid A^* on the state set X.

(c) Define the transition monoid $M(\mathcal{A})$ of an automaton $\mathcal{A}$. List all the elements of the monoid $M(\mathcal{A})$ where $\mathcal{A}$ is as in (a).

(d) Define the language $L(\mathcal{A})$ accepted by an automaton $\mathcal{A}$. What is $L(\mathcal{A})$ for $\mathcal{A}$ as in (a)?

p.t.o.

2.

(a) What is a monoid?

(b) Let M be a monoid. For $A, B \subseteq M$ define the product

$$AB = \{ab \mid a \in A,\ b \in B\} \subseteq M.$$

Show that the power set $2^M = \{A : A \subseteq M\}$ is a monoid with identity $\{1_M\}$.

(c) What is a homomorphism of monoids?

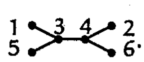
(d) Show that the map $f \colon M \to 2^M$ defined by $f(a) = \{a\}$ for $a \in M$ is an injective homomorphism of monoids.

3.

(a) Let G be a group acting on a set X. For $x \in X$ define the stabiliser G_x and the orbit $x.G$.

(b) Show that the distinct orbits $X/G = \{x.G \mid x \in X\}$ form a partition of X.

(c) Describe the orbit algorithm: what is its input, what is its output and what steps are performed?

(d) Let $G = \operatorname{Aut} \mathcal{G}$, where $\mathcal{G} =$ 

. Write down the elements of G. What are the orbits of G on $X = \{1, 2, 3, 4, 5, 6\}$?

4.

(a) Suppose the group G acts on the set X. Show that if $|G| = p^n$ for a prime p not dividing $|X|$ then there exists a point $x \in X$ such that $x.a = x$ for all $a \in G$.

(b) Suppose the group G has order $p^e m$ where p is a prime, e and m are positive integer and m is not divisible by p. Prove that G has a subgroup of order p^e.

(c) State the two other parts of Sylow's Theorem.

(d) Show that there is no simple group of order (i) 2004, (ii) p^n for a prime p and an integer $n > 1$.