GX 204

OLLSCOIL na hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY of IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004— HONOURS

THIRD ARTS and SCIENCE EXAMINATIONS

MATHEMATICS

MA385 and MA378 — NUMERICAL ANALYSIS

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Time allowed: *three* hours
Those taking one section should answer *three* questions
Those taking both sections should answer *five* questions, at least *two* from each section.

SECTION A — MA385

1. (a) Suppose that a real-valued function $g(x)$ is continuous and defined on $[a, b]$, and that $g(x) \in [a, b]$ for all $x \in [a, b]$. Show that $g(x)$ has a fixed point $\tau \in [a, b]$.
Suppose further that $g(x)$ is a *contraction* on $[a, b]$, i.e, there is a number $L \in (0, 1)$ such that

$$|g(\alpha) - g(\beta)| \leq L|\alpha - \beta| \text{ for all } \alpha, \beta \in [a, b]. \tag{1}$$

Show that

(i) the fixed point τ of $g(x)$ is unique.

(ii) the sequence defined by $x_0 \in [a, b]$ and $x_i = g(x_{i-1})$ for $i = 1, \ldots,$ converges to τ.

(b) Consider the problem: *Let* $f(x) = x^2 - x - 2$. *Find* $\tau \in [0, 2]$ *such that* $f(\tau) = 0$.
Taking $x_0 = 1$, and working to six significant figures, calculate x_4 for each of the following methods

(i) Fixed Point Iteration, with $g(x) = \sqrt{x + 2}$.

(ii) Newton's method.

Why does Newton's method converge faster?

2. (a) Explain what is meant by a *well-conditioned* function. Show that multiplication is well-conditioned, but addition is not.

(b) Define the terms *Unit Lower Triangular Matrix* and *Upper Triangular Matrix*.
Write down the LU-factorisation of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 9 \\ 4 & 9 & 1 \end{pmatrix}.$$

Hence solve $Ax = b$ where $b = (1, 1, 1)^T$.

3. (a) Given a norm on $\mathbb{R}^n$, the associated *subordinate matrix norm* is

$$\|A\|_\infty = \max_{v\in\mathbb{R}^n/\{0\}} \frac{\|Av\|_\infty}{\|v\|_\infty},$$

where $A \in \mathbb{R}^{n\times n}$. Show that

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |a_{ij}|.$$

Show that any subordinate matrix norm is *consistent*: $\|AB\| \le \|A\|\|B\|$, for $A, B \in \mathbb{R}^{n\times n}$.

(b) The *condition number of a matrix* is defined as $\kappa(A) = \|A\|\|A^{-1}\|$.
Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and suppose that $A\boldsymbol{x} = \boldsymbol{b}$ and $A(\boldsymbol{x} + \delta\boldsymbol{x}) = (\boldsymbol{b} + \delta\boldsymbol{b})$ where $\boldsymbol{b}, \boldsymbol{x} \in \mathbb{R}^n/\{0\}$ and $\delta\boldsymbol{b}, \delta\boldsymbol{x} \in \mathbb{R}^n$. Prove that

$$\frac{\|\delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \le \kappa(A)\frac{\|\delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}. \tag{2}$$

Suppose we are using a computer to solve the linear system: $A\boldsymbol{x} = b$ where

$$A = \begin{pmatrix} 10 & 12 \\ 0.08 & 0.1 \end{pmatrix}.$$

and, due to round-off error, right-hand side has a relative error (in the ∞-norm) of 10^{-6}. Use (2) to give an upper bound for the relative error in the computed solution.

4. (a) Define $R(v)$, the *Rayleigh Quotient* (associated with a given matrix $A \in \mathbb{R}^{n\times n}_{\text{sym}}$) of the vector v.
Suppose that A has eigenvalues $\lambda_{min} = \lambda_1 < \lambda_2 < \cdots < \lambda_n = \lambda_{max}$. Prove that, for any $v \in \mathbb{R}^n$.

$$\lambda_{min} \le R(v) \le \lambda_{max}.$$

(b) Consider the Initial Value Problem

$$y(x_0) = y_0 \quad \text{and} \quad y' = f(x, y) \text{ for } x \ge x_0.$$

Write down *Euler's Method* for this problem.
Suppose that $f(x, y)$ satisfies the Lipschitz condition $|f(x, u) - f(x, v)| \le L|u - v|$ for $L > 0$. Let T_n be the truncation error at step n, and $T = \max_{i=1,\dots n} |T_n|$. Show that the *global error* $e_n := y(x_n) - y_n$ satisfies

$$|e_n| \le |e_{n-1}|(1 - hL) + hT, \qquad n = 1, 0, \dots, N.$$

where $h = x_i - x_{i-1}$ for each i.
Hence deduce that

$$|e_n| \le \frac{T}{L}(\mathrm{e}^{L(x_n - x_0)} - 1), \qquad n = 1, 2, \dots, N.$$

5. (a) Suppose that $f(x)$ is a real-valued function such that $f^{(k)}(x)$ is continuous on $[a, b]$ for $k = 0, \ldots, n+2$. Let $p_n(x)$ be the polynomial of degree n that interpolates the function $f(x)$ at the distinct points $a = x_0, x_1, \ldots, x_n = b$.
Show that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\tau)}{n!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

for some $\tau \in (a, b)$.

(b) Write down the Lagrange form of $p_2(x)$, the polynomial of degree 2 that interpolates $f(x) = e^{-x}$ at $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.
Evaluate $p_2(x)$ at $x = 1.5$.
Give an upper bound for the error $|f(x) - p_2(x)|$ at $x = 1.5$.

6. (a) Define the *natural piecewise cubic (spline) interpolant* to the points (x_0, f_0), (x_1, f_1), $\ldots (x_n, f_n)$.
Suppose that $x_j - x_{j-1} = h = 1/n$ for $j = 1, \ldots, n$. Let $s_i(x)$ is the ith component of $s(x)$, with support only on $[x_{i-1}, x_i]$. Show that it can be written as

$$s_i(x) = \alpha_i(x - x_{i-1}) + \beta_i(x_i - x) + \frac{\sigma_{i-1}}{6h}(x_i - x)^3 + \frac{\sigma_i}{6h}(x - x_{i-1})^3,$$

where α_i and β_i are given by

$$\alpha_i = \frac{f_i}{h} - \frac{h}{6}\sigma_i, \qquad \beta_i = \frac{f_{i-1}}{h} - \frac{h}{6}\sigma_{i-1}.$$

Hence deduce that $s(x)$ can be determined from solving the system: $\sigma_1 = 0$, $\sigma_n = 0$ and

$$\frac{1}{6}(\sigma_{i-1} + 4\sigma_i + \sigma_{i+1}) = \frac{1}{h^2}(f_{i-1} - 2f_i + f_{i+1}) \quad \text{for} \quad i = 2, \ldots, n-1.$$

(b) Let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. Find the natural cubic spline that interpolates $f_0 = 0$, $f_1 = 2$ and $f_2 = 1$. What is its value at $x = 1.5$?

7. (a) Consider the 4-point Newton-Cotes Quadrature method:

$$Q(f) = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + a_3 f(x_3)$$

for approxmating $\int_0^1 f(x)dx$, with $x_i = i/3$. Using only that the method is exact for polynomials of degree 3 or less, deduce the values of the quadrature weights a_0, a_1, a_2 and a_3.
Also show how any one of the weights could be deduced by integrating an appropriate Lagrange Polynomial.

(b) Suppose that $Q_n(f)$ is a Newton-Cotes quadrature rule that is precise for polynomials of degree n or less. Show that, if n is even, it is in fact precise for any polynomial of degree $n + 1$ or less.

8. (a) Assume that $f(x)$ and is first four derivatives are continuous on $[x_{i-1}, x_{i+1}]$. Derive the second-order difference formula:

$$D^2(f_i) := \frac{1}{h^2}(f_{i-1} - 2f_i + f_{i+1}),$$

and the corresponding estimate for $|f''(x_i) - D^2(f_i)|$, where $x_i - x_{i-1} = x_{i+1} - x_i = h$.
Suppose that $|f^{(iv)}(x)| \leq M$ for some constant M and any $x_{i-1} \leq x \leq x_{i+1}$. Also suppose that

$$\hat{f}(x) = f(x) + e(x), \quad \text{and} \quad |e(x)| \leq \epsilon.$$

Find an expression for h in terms of M and ϵ that minimizes $|f''(x_i) - D^2(\hat{f}(x_i))|$.

(b) Use a finite difference method, based on the difference operator in Part (a), to find an approximate solution to

$$L(u) := -u''(x) + u(x) = 2 \quad \text{for} \quad x \in (0, 3),$$
$$u(0) = 0, u(3) = 1,$$

using four equally spaced mesh points.