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NATIONAL UNIVERSITY OF IRELAND, GALWAY

SEMESTER II EXAMINATIONS 2003 – 2004

MODULE CODE:MA 391
MODULE: PROBABILITY

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INSTRUCTIONS: **Answer the ten questions in PART A (30 marks) and two of the questions in PART B (35 marks each).**

DURATION: **Two hours**

PART A

[Multiple choice. 30 marks] In each of questions **A1.** through **A10.** below, **write down one** choice of answer. For example, if in **A1.** below you think **A)** is the answer, you would write in your answer book A1.A) .

A1. If $f(x)$ is the density function of a continuous random variable X, **what** must $\int_{-\infty}^{\infty} f(x)dx$ equal?

A) 0 B) 0.5 C) 1 .

A2. If $X \sim N(\mu, \sigma^2)$, **what** is the distribution of $Y = aX + b$ where a and b are non-zero constants?

A) $N(a\mu,\ a^2\sigma^2)$ B) $N(a\mu + b,\ a\sigma^2)$ C) $N(a\mu + b,\ a^2\sigma^2)$ D) $N(a\mu + b,\ a^2\sigma)$.

A3. Suppose that the marks of a large population of students have a normal distribution with mean $\mu = 50$ and standard deviation $\sigma = 10$. **What** proportion of students obtain marks between 40 and 70 ?

Note: If $Z \sim N(0,1)$, then $P(Z > 1) = 0.1587$, $P(Z > 2) = 0.0228$.

A) 1 B) 0.1587 C) 0.9772 D) 0.8185 E) 0.8413 F) 0.4772 .

A4. If X and Y are independent and each has mean 2 and variance 2, **what** is $E(XY^2)$?

A) 2 B) 4 C) 6 D) 8 E) 10 F) 12 .

A5. Discrete random variables X and Y have the joint density shown in the following table.

$Y\downarrow\ X\rightarrow$	1	2
1	$\frac{1}{10}$	$\frac{2}{10}$
2	$\frac{3}{10}$	$\frac{4}{10}$

What is the conditional density of Y given $X = 2$?

A) $f_{Y|X}(y|2) = \begin{cases} \frac{6}{10}, & y = 1 \\ \frac{4}{10}, & y = 2 \end{cases}$ B) $f_{Y|X}(y|2) = \begin{cases} \frac{4}{10}, & y = 1 \\ \frac{6}{10}, & y = 2 \end{cases}$ C) $f_{Y|X}(y|2) = \begin{cases} \frac{1}{3}, & y = 1 \\ \frac{2}{3}, & y = 2 \end{cases}$

D) $f_{Y|X}(y|2) = \begin{cases} \frac{1}{2}, & y = 1 \\ \frac{1}{2}, & y = 2 \end{cases}$ E) $f_{Y|X}(y|2) = \begin{cases} \frac{3}{4}, & y = 1 \\ \frac{1}{4}, & y = 2 \end{cases}$ F) $f_{Y|X}(y|2) = \begin{cases} \frac{1}{4}, & y = 1 \\ \frac{3}{4}, & y = 2 \end{cases}$.

A6. Let X denote the decay time of some radioactive particle and assume that X has the exponential density $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$. Suppose λ is such that $P(X \geq 0.01) = 1/2$. What number t satisfies $P(X \geq t) = 0.9$?

A) $exp\left(\frac{0.9}{2}\right)$ B) $exp\left(\frac{2}{0.9}\right)$ C) $\frac{-\ln(2)}{\ln(0.9)}$ D) $\frac{-\ln(0.9)}{100\ln(2)}$ E) $-\ln(0.9)$ F) $100\ln(2)$

A7. Two people get on an elevator at the ground floor, and there are five floors above the ground floor. The two people will get off the elevator independently and each person has a probability of 1/5 of getting off at any of the five floors above the ground floor. **What** is the expected number of stops that the elevator will make above the ground floor?

A) $\frac{9}{5}$ B) $\frac{18}{25}$ C) 1 D) $\frac{8}{5}$ E) 2 F) none of these.

A8. Let X be a nonnegative continuous random variable having density $f(x)$ and cumulative distribution function $F(x) := P(X \leq x)$, $-\infty < x < \infty$. Assume that $E(X) = \int_0^\infty xf(x)dx$ exists. **Which** one of the following is another expression for $E(X)$?

A) $\int_0^\infty F(x)dx$ B) $\int_0^\infty \ln F(x)dx$ C) $\int_0^\infty F(x)f(x)dx$

D) $\int_0^\infty [F(x)]^2 dx$ E) $\int_0^\infty \frac{1}{F(x)}dx$ F) $\int_0^\infty (1 - F(x))dx$.

A9. If $X \sim N(0,1)$, i.e. the density of X is $f(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, $-\infty < x < \infty$, **what** is $E(|X|)$?

A) $\frac{1}{\sqrt{2\pi}}$ B) 0 C) 1 D) $\sqrt{2\pi}$ E) $\frac{1}{\sqrt{\pi}}$ F) $\sqrt{\frac{2}{\pi}}$.

A10. Let $X_1, X_2, \ldots, X_{16}$ be a random sample of size 16 from $N(3,16)$. **What** is the distribution of $\left(\bar{X} - 3\right)^2$?

A) $N(2,9)$ B) $N(3,1)$ C) $N(2,3)$ D) χ^2_9 E) χ^2_8 F) χ^2_1.

PART B

B1. In some parts of this question use, whenever you need it, the following information concerning the Gamma function. For any $c > 0$ and $d > 0$, $\int_0^\infty u^{c-1}e^{-u/d}du = \Gamma(c)d^c$, $\Gamma(c) = (c-1)\Gamma(c-1)$, $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(a) Let X have the Gamma(α,β) density, i.e. $f_X(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, & x > 0 \\ 0, & \text{elsewhere,} \end{cases}$ $(\alpha > 0,$ $\beta > 0)$.

(i) *[7 marks]* **Show** that the moment generating function of X is

$$M(t) = 1/(1-\beta t)^\alpha,\ t < 1/\beta.$$

(ii) *[7 marks]* Accept that $P(X \geq r) \leq e^{-rt}M(t)$, for every r and all t, $0 < t < 1/\beta$. **Derive** a lower bound (depending only on α) for $P(X \geq 2\alpha\beta)$.

(b) Let $X \sim N(\mu,\sigma^2)$; thus the density of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left\{\frac{-1}{2\sigma^2}(x-\mu)^2\right\}, -\infty < x < \infty.$$

(i) *[6 + 1 = 7 marks]* Briefly **explain** why $E(X-\mu)^m = 0$ for m an odd positive integer. Specialize this result to **show** that the parameter μ in the above density is in fact $E(X)$.

(ii) *[6 + 1 = 7 marks]* For any even positive integer m, find an expression for $E(X-\mu)^m$ in terms of m, μ and σ^2. Specialize your result to **show** that the parameter σ^2 is in fact $Var(X)$.

(iii) *[7 marks]* Show that the random variable $Z = \dfrac{X-\mu}{\sigma}$ has the $N(0,1)$ distribution.

B2. Tom and Mary will arrive *independently* at a nightclub sometime after midnight (time 0). Tom's arrival time X and Mary's arrival time Y are each exponentially distributed with mean 1 hour:

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases},\quad f_Y(y) = \begin{cases} e^{-y}, & y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(i) *[2 marks]* **Explain** briefly why the above information implies that the joint density function of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x > 0,\ y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(ii) *[12 marks]* **Find** the density of $U = X - Y$.

(iii) *[7 + 7 = 14 marks]* Whoever arrives first will wait for at most 1 hour for the arrival of the other. **Find** the probability that they meet, firstly by using your result in (ii) above and secondly by integrating $f_{X,Y}(x,y)$ in (i) above over an appropriate region.

(iv) *[2 marks]* **Find** $E(X-Y)$.

(v) *[5 marks]* **Find** $P(U < 2 \mid U > 0)$.

B3.

(i) *[6 marks]* Let Y be a non-negative random variable. If $E(Y)$ exists, **show** that for each constant $c > 0$,

$$P(Y \geq c) \leq \frac{E(Y)}{c}.$$

(ii) *[6 marks]* Using the result in (i) above, **prove** Chebychev's inequality, i.e. if X has finite variance σ^2, then **prove** that for any $t > 0$, $P(|X - E(X)| \geq t) \leq \sigma^2/t^2$.

(iii) *[6 marks]* Let $X_1, X_2, \ldots, X_n$ be a random sample from an infinite population that has mean μ and variance σ^2, and define $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. **Show** that $E(\overline{X}) = \mu$ and $Var(\overline{X}) = \sigma^2/n$.

(iv) *[6 marks]* **State** the Weak law of Large Numbers (WLLN) and prove this law using Chebychev's inequality.

(v) *[6 marks]* **State**, without proof, any version of the Central Limit Theorem (CLT).

(vi) *[5 marks]* Use the results of (iii) and (v) above to **show** that if a large random sample of n Irish people is selected, the sample proportion $\hat{\theta}$ of smokers has an approximate normal distribution with mean θ and variance $\dfrac{\theta(1-\theta)}{n}$, where θ is the population proportion of smokers.

Hint: Let $X_i = \begin{cases} 1, & \text{if the } i\text{th selected person is a smoker} \\ 0, & \text{if the } i\text{th selected person is a non-smoker} \end{cases}, \; i = 1,2,\ldots,n.$

and then use without proof the results that $E(X_i) = \theta$ and $Var(X_i) = \theta(1-\theta)$.
