Gx2045

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

B.A. and B.Sc. Degree Examination

MATHEMATICS [MA418]

MA418 — *DIFFERENTIAL EQUATIONS WITH FINANCIAL DERIVATIVES*

Dr. Dave Johnson
Professor T. Hurley
Dr. J. Burns

Time allowed: *Two* hours.
Answer three questions.

Q1. (a) Define Standard Brownian motion $(B_t \,|\, t \geq o)$, and prove that it is a Gaussian process.

(b) Prove that $(T^{1/2}B_{t_1}, \ldots, T^{1/2}B_{t_n}) = (B_{Tt_1}, \ldots, B_{Tt_n})$ for every $T > 0$, and every choice of $t_i > 0$, $i = 1, \ldots, n$, and integer $n > 0$.

(c) Prove that $B_t^2 - t$ is an $\mathcal{F}_t$ martingale, where $\mathcal{F}_t$ is the natural filtration for Brownian motion.

Q2. (a) Define the *conditional expectation* $E(X|\mathcal{F})$ of a random variable X given a σ-field $\mathcal{F}$, and give a geometric interpretation of it. Prove that $E(X|\mathcal{F}) = E(E(X|\mathcal{F}')|\mathcal{F})$ if $\mathcal{F} \subset \mathcal{F}'$.

(b) Let $\Omega = [0, 1]$ with the σ- field of Borel sets, and let P be Legesgue measure on $[0, 1]$. Consider the random variables $X(\omega) = 2\omega^2$, $Y(\omega) = \omega(1 - \omega)$, $\omega \in [0, 1]$. Describe $\sigma(Y)$, and calculate $E(X|\sigma(Y))$.

p.t.o.

Q3. (a) State the Itô lemma for a process of the form $f(t, X_t)$, where $X_t = X_0 + \int_0^t A_s^{(1)} ds + \int_0^t A_s^{(2)} dB_s$, and both $A^{(1)}$ and $A^{(2)}$ are adapted to Brownian motion.

(b) Consider a self-financing strategy (a_t, b_t), and associated value process $V_t = a_t X_t + b_t \beta_t = u(T - t, X_t)$, $t \in [0, T]$. Assuming the stock X_t and bond β_t satisfy $dX_t = cX_t dt + \sigma X_t dB_t$, $c, \sigma > 0$, and $d\beta_t = r\beta_t dt$, $r > 0$, respectively, derive the Black-Scholes partial differential equation;

$$u_1(t,x) = \frac{1}{2}\sigma^2 x^2 u_{22}(t,x) + rxu_2(t,x) - ru(t,x)$$

$x > 0, t \in [0, T]$.

Q4. (a) State Girsanov's (change of measure) theorem. Assume that in the Black-Scholes model there exists a self-financing strategy (a_t, b_t) such that the value V_t of the portfolio at time t is $V_t = a_t X_t + b_t \beta_t$, $t \in [0, T]$, and V_T is equal to the contingent claim $h(X_T)$. Prove that

$$V_t = E_Q[e^{-r(T-t)} h(X_T) | \mathfrak{F}_t]$$

where $\mathfrak{F}_t$ is the natural filtration for Brownian motion, and Q is the equivalent martingale measure in Girsanov's theorem which converts $B_t + \frac{(c-r)}{\sigma} t$ into standard Brownian motion.

(b) Prove that the European put price P_T and the European call price C_T (with strike price K) are related by the formula $P_T = C_T - X_0 + K$ where the stock price satisfies $dX_t = cX_t dt + \sigma X_t dB_t$, $c, \sigma > 0$.