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OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

B.Sc. (Part II) EXAMINATION

HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS [MA482]

MA482 – Functional Analysis

Dr. D. Johnson

Professor T. Hurley

Dr. P. Kirwan

Time Allowed: *Two* hours

Full marks for *three* questions

Question 1

(a) Show that $\|(x_1, x_2)\|_{\frac{1}{2}} = (\sqrt{x_1} + \sqrt{x_2})^2$ does not define on a norm on $\mathbf{R}^2$.

(b) State and prove the Cauchy–Schwartz inequality.

(c) Let B_X be the closed unit ball of a normed space X. Show that;

(i) B_X is convex.

(ii) $\|x\| = \inf\{\lambda > 0 : x \in \lambda B_X\} \qquad \forall x \in X$.

P.T.O.

Question 2

(a) Show that if X is a finite dimensional normed space then every linear operator from X to X is bounded.

(b) Let X be the normed space of all polynomials on $[0,1]$ with norm $\|p\| = \max\{|p(t)|: 0 \le t \le 1\}$. Show that the differentiation operator

$$T(p)(t) = p'(t)$$

is an unbounded operator from X to X.

(c) Let $X = \mathbf{R}^2$ with the ℓ_2 norm and $Y = \mathbf{R}^2$ with the ℓ_2 norm and let $T: X \to Y$ be given by

$$T((x_1, x_2)) = (x_1, x_1 + x_2).$$

Determine the norm of T.

Question 3

(a) Let e_n be the sequence where the n–th term is 1 and all other terms are 0. Show that (e_n) is a Schauder basis for c_0 but not for ℓ_∞.

(b) Show that the dual space of c_0 is ℓ_1.

(c) Let X, Y be normed spaces and $T \in \mathcal{L}(X, Y)$.

(i) Define the adjoint T^* of T.

(ii) Show that $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

Question 4

(a) Let A be a closed convex subset of a Hilbert space H and $x \in H$. Show that there exists a unique point in A that is closest to x, i.e. $\exists y \in A$ such that

$$\|x - y\| = \inf_{z \in A} \|x - z\|$$

(b) Determine the Fourier series expansion of the function $f(t) = t$ in the interval $[-\pi, \pi]$.

(c) Show that $(C[0,1], \|\cdot\|_\infty)$ is not an inner product space.