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OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS — [MA490/482]

MEASURE THEORY AND FUNCTIONAL ANALYSIS

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Time allowed: **Three** hours.
Full marks for five questions.
Use separate answer books for each section.

SECTION A — *MEASURE THEORY*

A1. Answer *four* of the following:

(a) In Bernoulli space, let E be the event that the sequence of coin tosses never produces the sequence HTHT. Show that E belongs to the σ-algebra generated by the simple events E_n. (E_n is the event that the nth toss is H.)

(b) Give a description of the elements of the Cantor set C and explain briefly why this set is uncountable.

(c) Let $A_n = (1 + 1/n, 3 + 1/n)$ if n is odd and $(0, 2)$ if n is even. Find the sets $\limsup_n A_n$ and $\liminf_n A_n$.

(d) What can you say about the σ-algebra of measurable subsets of X if
(i) only the constant functions on X are measurable?
(ii) every function on X is measurable?

(e) Show that if the function f is measurable then so is $|f|$. Give an example of a non-measurable function f such that $|f|$ is measurable.

(f) Let f be $\chi_{\mathbb{Q}}$, the characteristic function of the set of rational numbers. Explain why f is Lebesgue integrable on the interval $[0, 1]$ and find its integral. Explain briefly why f is not Riemann integrable on $[0, 1]$.

p.t.o.

A2. (a) Give the definition of an *algebra* and a *σ-algebra*, and give an example of an algebra that is not a σ-algebra. Show that the Cantor set C belongs to the σ-algebra of subsets of $\mathbf{R}$ generated by the intervals.

(b) Give the definition of a measure. If μ is a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X and (E_n) is a decreasing sequence of sets in $\mathcal{A}$ with intersection E and if $\mu(E_1) < \infty$, show that

$$\mu(E) = \lim_{n\to\infty} \mu(E_n)\,.$$

Show by an example that this result may fail if the condition $\mu(E_1) < \infty$ is omitted.

A3. (a) Give the definition of a *measurable function.* Show that if f and g are measurable then so is the function $f + g$.

(b) Let (f_n) be a sequence of measurable functions. Show that the functions $\sup_n f_n$ and $\inf_n f_n$ are measurable.

(c) Let f be a nonnegative measurable function. Prove that there is an increasing sequence of measurable simple functions that converges pointwise to f. Show that if f is bounded, then the convergence is uniform.

A4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account (without proofs) of the definition of the integral $\displaystyle\int_X f\,d\mu$, starting with the integral of a simple measurable function.

(b) If f is a measurable function on X, show that f is integrable if and only if

$$\int_X |f|\,d\mu < \infty\,.$$

(c) State (without proof) two convergence theorems that allow term by term integration of a sequence or series of integrable functions. Give an example of a sequence (f_n) of integrable functions that converges pointwise to an integrable function f, but is such that the integrals $\int_X f_n\,d\mu$ do not converge to $\int_X f\,d\mu$.

p.t.o.

SECTION B — *FUNCTIONAL ANALYSIS*

B1. (a) Show that $\|(x_1, x_2)\|_{\frac{1}{2}} = (\sqrt{x_1} + \sqrt{x_2})^2$ does not define on a norm on $\mathbf{R}^2$.

(b) State and prove the Cauchy–Schwartz inequality.

(c) Let B_X be the closed unit ball of a normed space X. Show that;

(i) B_X is convex.

(ii) $\|x\| = \inf\{\lambda > 0 : x \in \lambda B_X\} \qquad \forall x \in X$.

B2. (a) Show that if X is a finite dimensional normed space then every linear operator from X to X is bounded.

(b) Let X be the normed space of all polynomials on $[0, 1]$ with norm $\|p\| = \max\{|p(t)|: 0 \leq t \leq 1\}$. Show that the differentiation operator

$$T(p)(t) = p'(t)$$

is an unbounded operator from X to X.

(c) Let $X = \mathbf{R}^2$ with the ℓ_2 norm and $Y = \mathbf{R}^2$ with the ℓ_2 norm and let $T: X \to Y$ be given by

$$T((x_1, x_2)) = (x_1, x_1 + x_2).$$

Determine the norm of T.

B3. (a) Let e_n be the sequence where the n–th term is 1 and all other terms are 0. Show that (e_n) is a Schauder basis for c_0 but not for ℓ_∞.

(b) Show that the dual space of c_0 is ℓ_1.

(c) Let X, Y be normed spaces and $T \in \mathcal{L}(X, Y)$.

(i) Define the adjoint T^* of T.

(ii) Show that $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

B4. (a) Let A be a closed convex subset of a Hilbert space H and $x \in H$. Show that there exists a unique point in A that is closest to x, i.e. $\exists\ y \in A$ such that

$$\|x - y\| = \inf_{z \in A} \|x - z\|$$

(b) Determine the Fourier series expansion of the function $f(t) = t$ in the interval $[-\pi, \pi]$.

(c) Show that $(C[0, 1], \|\cdot\|_\infty)$ is not an inner product space.