GX2049

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS — [MA490]

MEASURE THEORY

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Time allowed: **Two** hours.
Answer *three* questions.

1. Answer *four* of the following:

(a) In Bernoulli space, let E be the event that the sequence of coin tosses never produces the sequence HTHT. Show that E belongs to the σ-algebra generated by the simple events E_n. (E_n is the event that the nth toss is H.)

(b) Give a description of the elements of the Cantor set C and explain briefly why this set is uncountable.

(c) Let $A_n = (1 + 1/n, 3 + 1/n)$ if n is odd and $(0, 2)$ if n is even. Find the sets $\limsup_n A_n$ and $\liminf_n A_n$.

(d) What can you say about the σ-algebra of measurable subsets of X if
(i) only the constant functions on X are measurable?
(ii) every function on X is measurable?

(e) Show that if the function f is measurable then so is $|f|$. Give an example of a non-measurable function f such that $|f|$ is measurable.

(f) Let f be $\chi_{\mathbb{Q}}$, the characteristic function of the set of rational numbers. Explain why f is Lebesgue integrable on the interval $[0, 1]$ and find its integral. Explain briefly why f is not Riemann integrable on $[0, 1]$.

p.t.o.

2. (a) Give the definition of an *algebra* and a *σ-algebra*, and give an example of an algebra that is not a σ-algebra. Show that the Cantor set C belongs to the σ-algebra of subsets of $\mathbf{R}$ generated by the intervals.

(b) Give the definition of a measure. If μ is a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X and (E_n) is a decreasing sequence of sets in $\mathcal{A}$ with intersection E and if $\mu(E_1) < \infty$, show that

$$\mu(E) = \lim_{n\to\infty} \mu(E_n).$$

Show by an example that this result may fail if the condition $\mu(E_1) < \infty$ is omitted.

3. (a) Give the definition of a *measurable function.* Show that if f and g are measurable then so is the function $f + g$.

(b) Let (f_n) be a sequence of measurable functions. Show that the functions $\sup_n f_n$ and $\inf_n f_n$ are measurable.

(c) Let f be a nonnegative measurable function. Prove that there is an increasing sequence of measurable simple functions that converges pointwise to f. Show that if f is bounded, then the convergence is uniform.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account (without proofs) of the definition of the integral $\displaystyle\int_X f\, d\mu$, starting with the integral of a simple measurable function.

(b) If f is a measurable function on X, show that f is integrable if and only if

$$\int_X |f|\, d\mu < \infty.$$

(c) State (without proof) two convergence theorems that allow term by term integration of a sequence or series of integrable functions. Give an example of a sequence (f_n) of integrable functions that converges pointwise to an integrable function f, but is such that the integrals $\int_X f_n\, d\mu$ do not converge to $\int_X f\, d\mu$.