GX2051

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004

FOURTH UNIVERISTY EXAMINATION

STOCHASTIC PROCESSES - MA494

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Time allowed: *Two* hours.

In addition to this paper you should have available an electronic calculator not capable of storing text and logarithmic tables.

ATTEMPT THREE QUESTIONS.
ALL QUESTIONS HAVE EQUAL MARKS.

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1. (a) Define the Markov property mathematically when applied to a discrete parameter stochastic process.

(b) If a two state Markov chain has an initial distribution $\pi_0(x)$ and

$$P(X_{n+1} = 0 \,|X_n = 1) = p$$
$$P(X_{n+1} = 1 \,|X_n = 0) = q.$$

(i) Write down the transition matrix.

(ii) Find
$$P(X_0 = 1,\ X_1 = 1,\ X_2 = 1,\ X_3 = 1).$$

(c) Show that

$$P(X_0 = x_0, ..., X_n = x_n) = \pi_0(x_0).p(x_0, x_1)....p(x_{n-1}, x_n),$$

where $p(a, b)$ is the transition function for the process.

(d) Consider the Ehrenfest chain with $d = 4$ i.e. the process with two boxes labelled 1 and 2. There are d balls labelled 1 to d inclusive. Some/none of these balls are placed in box 1 and the remainder in box 2. An integer is picked at random between 1 and d inclusive and the ball with that label is removed from its box and replaced in the other box. Let X_n denote the number of balls in box 1 after the nth trial.

(i) Find the transition function $p(x, y)$.

(ii) Find the one and two step transition matrices.

(iii) Given π_0 to be a uniform distribution, find π_2.

(iv) Compute $P_2(T_{\{1,2\}} = 2)$.

pto

2. (a) (i) Define an absorption probability.

(ii) Suppose the set H_T of transient states is finite and let C be an irreducible closed set of recurrent states. Show that the system of equations

$$f(x) = \sum_{y \in C} p(x,y) + \sum_{y \in H_T} P(x,y) f(y),$$

for $x \in H_T$ has a unique solution $f(x) = \rho_C(x)$ where $x \in H_T$.

(b) Consider a Markov chain on {a, b, c, d, e, f} having transition matrix:

$$\begin{array}{c|cccccc}
 & a & b & c & d & e & f \\
\hline
a & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\
b & 0 & \frac{1}{5} & \frac{1}{5} & \frac{3}{5} & 0 & 0 \\
c & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\
d & 0 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 \\
e & 0 & 0 & 0 & 0 & 1 & 0 \\
f & \frac{1}{10} & \frac{1}{10} & 0 & \frac{1}{10} & 0 & \frac{7}{10}
\end{array}$$

(i) Determine which states are transient and which states are recurrent. Explain your answer fully.

(ii) Find $\rho_{\{b,c,d\}}(x)$ with $x = a, e, f$.

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3. (a) Show that a Markov chain having state space $\{0, ..., d\}$ and transition function p such that

$$\sum_{y=0}^{d} yp(x,y) = x \ , \ x = 0, ...d,$$

has the following properties

(i)

$$E[X_{n+1}|X_n = x] = x,$$

(ii)

$$E[X_n] = E[X_{n-1}] = ... = E[X_0].$$

(b) (i) Consider a Markov chain having transition function p such that $p(x,y) = \alpha(y)$, $x \in H$ and $y \in H$ where $\alpha(y)$ is constant for a particular y. Show that the chain has a unique stationary distribution π, given by $\pi(y) = \alpha(y)$, $y \in H$.

(ii) Consider a Markov chain having state space 0, 1, 2 and transition matrix

	0	1	2
0	.4	.4	.2
1	.3	.4	.3
2	.2	.4	.4

Show that this chain has a unique stationary distribution π and find π.

(iii) Let π be a stationary distribution of a Markov chain. Suppose y and z are two states such that for some constant c

$$p(x,y) = cp(x,z).$$

Show how to obtain and give the relationship between $\pi(y)$ and $\pi(z)$.

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4. (a) In the context of Markov pure jump processes

   (i) define and give an example of a process that explodes.

   (ii) define $F_x(t)$ and the transition probability Q_{xy}.

(b) The forward equation relating the probability that a Markov pure jump process starting at x will be at state y at time t to the infinitesimal parameters is

$$p'_{xy}(t) = \sum_z p_{xz}(t) q_{zy}.$$

The infinitesimal parameters q_{xy} are

$$q_{xy} = \begin{cases} -q_x & y = x \\ q_x Q_{xy} & y \neq x. \end{cases}$$

Find the transition function of the two-state birth and death process by solving the forward equation.