Gx2052

OLLSCOIL NA hÉIREANN, GAILLIMH

NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2004

Second Arts, Engineering and Science examinations

MM246 – Numerical Analysis

Dr. Dave Johnson
Prof. T. Hurley
Dr. E. Sköldberg

Time allowed: *two* hours.
Answer *three* questions.

1. (a) Use the composite Simpson's method with $n = 8$ to estimate

$$\int_0^1 e^{x^2}\,dx$$

(b) By calculating upper and lower bounds for the error in the estimate in (a), give an interval in which you can be certain that the exact value of the integral lies.

(c) Estimate the integral in (a) again with the composite Simpson's method with $n = 4$. Hence use Richardson extrapolation to get a better estimate.

2. Consider the following system of linear equations:

$$\begin{cases} 10x_1 & -x_2 & & = & 8 \\ -x_1 & +10x_2 & -2x_3 & = & 7 \\ & -2x_2 & +10x_3 & = & 6 \end{cases}$$

(a) Write down the Jacobi and Gauß-Seidel iterative schemes for this system.

(b) Working to four decimal places, perform three iterations of each of these schemes.

(c) Show that the Jacobi scheme converges for this system.

(d) How many iterations of the Jacobi scheme would be needed to get an estimate with error $< 10^{-5}$?

3. Let A be the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 4 & -1 \\ -1 & -1 & -5 \end{pmatrix}$$

(a) Are there any non-real eigenvalues? (Motivate your answer.)

(b) State Gershgorin's circle theorem and use it to find intervals which contain the eigenvalues of A.

(c) Carry out three iterations of the power method with initial vector $(1, 1, 1)^t$.

(d) Describe how the power method can be modified to find the eigenvalue of a matrix closest to a given number.

4. (a) Justify the improved Euler formula

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f\left(x_i + h, y_i + hf(x_i, y_i)\right)\right]$$

for finding approximations to the solution of a first-order initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

(b) Consider the following second order initial value problem:

$$x^2y'' - 2xy' - 2y = 0, \quad y(1) = y'(1) = 1$$

i. Write this as a system of first order initial value problems.

ii. Use the improved Euler method with $x_0 = 1$ and $h = 0.5$ to find an estimate of $y(2)$ and $y'(2)$.

A Let $J = \int_a^b f(x)\,dx$, with $x_i = a + ih$, $f_i = f(x_i)$ for $i \geq 0$, and $h = (b-a)/n$.

Trapezoid rule $J = J_{T,n} + \mathcal{E}_{T,n}$, where

$$J_{T,n} = \frac{h}{2}\left(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n\right)$$

and

$$\mathcal{E}_{T,n} = -\frac{h^2}{12}(b-a)f''(\xi), \quad \text{for some } \xi \in [a,b].$$

Simpson's rule (With an *even* number n of evenly spaced points) $J = J_{S,n} + \mathcal{E}_{S,n}$, where

$$J_{S,n} = \frac{h}{3}\left(f_0 + 4\sum_{i=1}^{n/2} f_{2i-1} + + \sum_{i=1}^{n/2-1} f_{2i} + f_n\right)$$

and

$$\mathcal{E}_{S,n} = -\frac{h^4}{180}(b-a)f^{(4)}(\xi), \quad \text{for some } \xi \in [a,b].$$

B Suppose $y(x)$ satisfies the first order initial value problem $y' = f(x,y)$, $y(x_0) = y_0$. Let $x_{i+1} = x_i + h$ for $i \geq 0$.

Improved Euler method

$$y_{i+1} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i)).\right]$$

Modified Euler method

$$y_{i+1} = y_i + hf\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}f(x_i, y_i)\right).$$