GX2053

OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS, 2004

THIRD ARTS, ENGINEERING & SCIENCE EXAMINATIONS

APPLIED MATHEMATICAL SCIENCE [AS300]

MM355 - NUMERICAL ANALYSIS

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Time allowed: *Three* hours.

Those answering both sections should attempt *two* questions from Section A and *two* questions from Section B.

Those answering one Section should attempt *three* questions.

PLEASE USE SEPARATE ANSWER BOOKS FOR EACH SECTION.

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SECTION A

1. (a) Find the spectral radius $\rho(B)$ of

$$B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

(b) Find $||C||_1$ and $||C||_\infty$ where

$$C = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

(c) If A is any square $n \times n$ matrix, X is a real column vector of length n and $||.||$ is a norm, prove that

$$||AX|| \leq ||A||.||X||$$

(d) Consider the linear system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 2 \\ x_1 + x_2 - 2x_3 &= -1 \end{aligned}$$

(i) With $x^{(0)} = (0, 0, 0)^t$, perform one iteration of the Gauss-Seidel method.

(ii) With $x^{(0)} = (0, 0, 0)^t$, perform one iteration of the Successive Over Relaxation (SOR) method, with $\omega = 1.2$

2. (a) State Gerschgorin's Circle Theorem and use it to separate the eigenvalues of the real symmetric matrix

$$\begin{pmatrix} 9 & 0 & 1 & 1 \\ 0 & -8 & 2 & -1 \\ 1 & 2 & 15 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

That is, find r disjoint regions, each containing precisely one of the eigenvalues (where r is the number of eigenvalues).

(b) (i) Define what it means for two square $n \times n$ matrices to be similar.

(ii) If A and B are similar matrices then show that they have the same eigenvalues.

(iii) If A and B are similar, then what is the relationship between the eigenvectors of A and the eigenvectors of B?

3. (a) For the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 3 & -1 \\ 8 & 4 & 2 \end{pmatrix}$$

find an orthogonal matrix Q such that QA is upper triangular. Hence find a QR factorisation of A.

(b) For the symmetric matrix

$$B = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 1 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

use Householder's Method to find a tridiagonal matrix with the same eigenvalues as B.

4. (a) The fourth order Adam's method applied to $y' = f(t, y)$ uses the formula

$$y_{i+1}^{(0)} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

as predictor and uses

$$y_{i+1}^{(1)} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2})$$

as corrector.

Consider the initial value problem

$$y' = y - 2t^2, \qquad 0 \leq t \leq 1$$

$$y(0) = 1$$

with step size $h = 0.1$.

If we assume that $y_1 = y(0.1) = 1.1$, $y_2 = y(0.2) = 1.2$ and $y_3 = y(0.3) = 1.3$, then use the fourth order Adams predictor/corrector method to estimate $y_4 = y(0.4)$.

(b) Show that the finite difference method for solving the boundary value problem

$$y'' = 2y' + 20y + t, \qquad 0 \leq t \leq 1$$

$$y(0) = 1, \quad y(1) = 3$$

with $N = 3$ reduces to solving

$$\begin{pmatrix} 13 & -3 & 0 \\ -5 & 13 & -3 \\ 0 & -5 & 13 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{79}{16} \\ -\frac{1}{8} \\ \frac{141}{16} \end{pmatrix}$$

Section B

5. Consider the following two point boundary value problem for a function $u = u(x)$:

$$-\frac{d^2u}{dx^2} - u = -x^2 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } x = 1.$$

a. Show that the weak form of this boundary value problem is given by

$$B(u,w) = l(w)$$

where

$$B(u,w) = \int_0^1 \left\{ \frac{du}{dx}\frac{dw}{dx} - uw \right\} dx \text{ and } l(w) = -\int_0^1 x^2 w \, dx$$

and where $w = w(x)$ is a sufficiently smooth weight function. In your derivation you should identify the primary variable and the secondary variable, as well as classifying the boundary conditions.

b. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^N$ are pre-selected approximation functions, show that for the Rayleigh-Ritz procedure, the $\{c_i\}_{i=1}^N$ are determined by solving the linear equations

$$\sum_{j=1}^{N} B_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N \text{ where } B_{ij} = B(\phi_i,\phi_j),\ F_i = l(\phi_i) - B(\phi_i,\phi_0).$$

c. Calculate the Rayleigh-Ritz approximation to the above problem for the case $N = 2$ with

$$\phi_0(x) = 0,\ \phi_1(x) = x(1-x),\ \phi_2(x) = x^2(1-x).$$

6. Consider the following two point boundary value problem for a function $u = u(x)$:

$$A(u) = f(x),\ 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 0 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + 9u,\ f(x) = 3.$$

a. Seeking an approximate solution to this problem of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^N$ are pre-selected approximation functions, show that for the Galerkin method, the $\{c_i\}_{i=1}^N$ are determined by solving the linear equations

$$\sum_{j=1}^{N} A_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N$$

where

$$A_{ij} = \int_0^1 \phi_i A(\phi_j)dx,\ F_i = \int_0^1 \phi_i(f - A(\phi_0))dx.$$

b. Calculate the Galerkin approximation to the above problem for the case $N = 2$ with

$$\phi_0(x) = 0,\ \phi_1(x) = \sin(\pi x/2),\ \phi_2(x) = \sin(3\pi x/2).$$

7. Consider the following two point boundary value problem for a function $u = u(x)$:

$$A(u) = f(x),\ 0 < x < 1,$$

$$u = 1 \text{ on } x = 0 \text{ and } \frac{du}{dx} = 2 \text{ on } x = 1,$$

where

$$A(u) = -\frac{d^2u}{dx^2} + xu,\ f(x) = x.$$

a. An approximate solution to this problem is sought of the form

$$U_N(x) = \phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots\ldots + c_N\phi_N(x)$$

where the $\{\phi_i(x)\}_{i=1}^{N}$ are pre-selected approximation functions. The residue, R, is defined by $R = A(U_N) - f$. Show that by minimizing $\int_0^1 R^2 dx$ with respect to the c_j's, $j = 1,2,\ldots,N$, that (the least-squared method)

$$\sum_{j=1}^{N} A_{ij}c_j = F_i \text{ for } i = 1,2,\ldots,N$$

where

$$A_{ij} = \int_0^1 A(\phi_i)A(\phi_j)dx,\ F_i = \int_0^1 A(\phi_i)(f - A(\phi_0))dx.$$

b. Calculate the least-squared approximation to the above problem for the case $N = 1$ with

$$\phi_0(x) = 1 + 2x,\ \phi_1(x) = x(x-2).$$

8. Consider the following two point boundary value problem for a function $u(x)$:

$$-\frac{d^2u}{dx^2} + u = 1 \text{ for } 0 < x < 1,$$

$$u = 0 \text{ on } x = 0 \text{ and } x = 1.$$

The interval $0 < x < 1$ is split into four equal subintervals of length 1/4 with a view to estimating the solution using the Finite Element Method.
The elements are denoted by $(x_{1,e}, x_{2,e})$ with $e = 1,2,3,4$. Estimating u in the interval $(x_{1,e}, x_{2,e})$ by the linear approximation

$$U_e = u_{1,e}\psi_{1,e}(x) + u_{2,e}\psi_{2,e}(x)$$

where $\psi_{1,e}(x) = 4(x_{2,e} - x)$, $\psi_{2,e}(x) = 4(x - x_{1,e})$, the element equations are given by (you are not required to show this) :

$$\sum_{j=1}^{2} K_{ij,e}u_{j,e} - f_{i,e} - (-1)^i Q_{i,e} = 0 \text{ for } i = 1,2 \text{ and } e = 1,2,3,4,$$

where

$$K_{ij,e} = \int_{x_{1,e}}^{x_{2,e}} \left(\frac{d\psi_{i,e}}{dx}\frac{d\psi_{j,e}}{dx} + \psi_{i,e}\psi_{j,e} \right) dx,\ f_{i,e} = \int_{x_{1,e}}^{x_{2,e}} \psi_{i,e}dx \text{ for } i,j = 1,2 \text{ and } e = 1,2,3,4,$$

$$Q_{1,e} = \left(\frac{du}{dx}\right)_{x=x_{1,e}},\ Q_{2,e} = \left(\frac{du}{dx}\right)_{x=x_{2,e}} \text{ for } e = 1,2,3,4.$$

a. Show that

$$\mathbf{k}^e = (K_{ij,e}) = \begin{pmatrix} 49/12 & -95/24 \\ -95/24 & 49/12 \end{pmatrix},\ \mathbf{f}^e = (f_{i,e}) = \begin{pmatrix} 1/8 \\ 1/8 \end{pmatrix} \text{ for } e = 1,2,3,4.$$

(cont'd overleaf)

Q.8. cont'd:

b. Writing $u_{2,1} = u_{1,2} = U_2$, $u_{2,2} = u_{1,3} = U_3$, $u_{2,3} = u_{1,4} = U_4$, show that

$$\begin{pmatrix} 49/6 & -95/24 & 0 \\ -95/24 & 49/6 & -95/24 \\ 0 & -95/24 & 49/6 \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}.$$

9. a. Consider the following initial value problem for a nonlinear system of ordinary differential equations:

$$\frac{dx}{dt} = 10(y-x),\ \frac{dy}{dt} = 28x - y - xz,\ \frac{dz}{dt} = xy - 8z/3,$$
$$x(0) = -10,\ y(0) = 20,\ z(0) = -5.$$

i. Write MAPLE commands to define this system of differential equations and initial conditions.

ii. Write MAPLE commands to obtain numerical estimates of $(x(t), y(t), z(t))$ at $t = 3$ and $t = 6$.

iii. Write MAPLE commands to plot a numerical solution of the above initial value problem in the (x,z) plane from $t = 0$ to $t = 30$ using 2000 points.

b. Write MAPLE commands to solve each of the following problems:

i. $\frac{d^4y}{dx^4} - y = 0, y(0) = 0, \frac{dy}{dx}(0) = 0, y(1) = 1, \frac{dy}{dx}(1) = 1.$

ii. $3u + v = 3,\ u - 2v - w = 0,\ u + v + w = 1.$

iii. $\frac{dy}{dt} = -y^3, y(0) = 1.$