Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

GX 2140

Christmas Examinations, 2003/2004

Exam Code(s)	2BS1, 2CA1, 2EL1, 2ER1, 2MR1, 2PT1, 3BS3, 3BS9, 3CS1, 3EL1, 3EL2, 3ER3, 4BS3.
Exam(s)	**Second Science and Arts**
Module Code(s)	MP205
Module(s)	Methods of Mathematical Physics(Pass)
Paper No.	
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Dr. M. S. Ó Confhaola;
	Dr. B. Gleeson.
Instructions:	Attempt *THREE* questions.
Duration	*TWO* hours
No. of Answer books	___________
Requirements:	___________
Handout	___________
MCQ	___________
Statistical Tables	YES, LOG TABLES
Graph Paper	___________
Log Graph Paper	___________
Other Material	___________
No. of Pages	3 PAGES (Excluding Cover Page)
Department(s)	MATHEMATICAL PHYSICS

1. (a) Prove the *Linearity Property* for Laplace transforms

$$\mathrm{L}\left[\alpha f(t)+\beta g(t)\right]=\alpha\overline{f(s)}+\beta\overline{g(s)},$$

where α, β are arbitrary constants.

(b) Prove the *First Shift theorem* for Laplace transforms

$$\mathrm{L}\left[e^{at}f(t)\right]=\overline{f(s-a)},$$

where a is an arbitrary constant.

(c) Use Laplace transforms to solve the initial value problem

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y(t)=6,$$

subject to

$$y(0)=0,\quad \dot{y}(0)=0.$$

2. (a) Calculate the inverse Laplace transform below using, the method of partial fractions and the Convolution Theorem:

$$\mathrm{L}^{-1}\left[\frac{3}{(s-1)(s+2)}\right].$$

Verify your answers by Laplace transforming the resulting function.

(b) Solve the following boundary value problem (BVP):

$$\frac{d^2y}{dt^2}-3\frac{dy}{dt}+2y(t)=0,$$

subject to the boundary conditions $y(0)=0$, $y(1)=e^2-e$.

3. The periodic function $f(x)$ is defined, on the interval $-\pi\leqslant x<\pi$, by

$$f(x)=x,$$

and elsewhere by $f(x+2\pi)=f(x)$.

(a) Find the Fourier series of $f(x)$.

(b) Sketch the function $f(x)$ on the interval $-3\pi\leqslant x<3\pi$.

(c) State the value to which the series converges at $x=\pi/2$. Hence prove that:

$$\frac{\pi}{4}=\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$$

4. (a) Show that the following pair of functions

$$u(x,y) = y^2 - x^2 \quad \text{AND} \quad v(x,y) = -2xy$$

satisfy the pair of partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{AND} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence or otherwise, show that these functions also satisfy

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{AND} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

(b) If the cartesian coordinates x and y are related to the plane polar coordinates r and θ by $x = r\cos(\theta)$ and $y = r\sin(\theta)$, prove that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2.$$

5. (a) Find all of the stationary points of the function

$$f(x,y) = x^3 + y^3 + 3x^2 + 6y^2 - 9x + 9y + 1,$$

and investigate the nature of each of these points.

(b) Use the method of Lagrange Multipliers to find the points on the curve, described by

$$x^2 + 8xy + 7y^2 = 180$$

which are closest to, and/or farthest from the origin $(0,0)$.

Table of Laplace Transforms

In all cases herein, a is a constant, and n is a positive integer:

$f(t) = \mathrm{L}^{-1}\left[\overline{f(s)}\right]$	$\overline{f(s)} = \mathrm{L}[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\mathrm{H}(t-a)$	$\frac{\exp[-as]}{s}$
$\delta(t-a)$	$\exp[-as]$

The Heaviside function, $\mathrm{H}(t-a)$, is defined by

$$\boxed{\mathrm{H}(t-a) = \begin{cases} 0 & \text{FOR } 0 \leqslant t < a, \\ 1 & \text{FOR } t \geqslant a. \end{cases}}$$

(In this context, the constant a is understood to be positive).