GX 2141

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

Christmas Examinations, 2003/2004

Exam Code(s)	2BS1, 2CS1, 2EL1, 2PT1, 3CS1.
Exam(s)	**Second Science and Arts**
Module Code(s)	MP207
Module(s)	Methods of Mathematical Physics(Honours)
Paper No.	
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan;
Internal Examiner(s)	Dr. M. S. Ó Confhaola;
	Dr. B. Gleeson;
Instructions:	Attempt *THREE* questions.
Duration	*TWO* hours
No. of Answer books	
Requirements:	
Handout	
MCQ	
Statistical Tables	YES, LOG TABLES
Graph Paper	
Log Graph Paper	
Other Material	
No. of Pages	3 PAGES (Excluding Cover Page)
Department(s)	MATHEMATICAL PHYSICS

1. (a) Prove the *'Differential' theorem* for Laplace transforms

$$\mathrm{L}\left[t\, f\left(t\right)\right] = -\frac{d}{ds}\left(\mathrm{L}\left[f\left(t\right)\right]\right).$$

(b) Prove the *Second Shift theorem* for Laplace transforms

$$\mathrm{L}\left[\mathrm{H}\left(t-a\right) f\left(t-a\right)\right] = e^{-as}\,\mathrm{L}\left[f\left(t\right)\right],$$

where a is an arbitrary positive constant.

(c) Use Laplace transforms to solve the following set of simultaneous differential equations

$$\frac{dx}{dt} + 4y\left(t\right) = \cos\left(3t\right),$$
$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + 4x\left(t\right) = \sin\left(3t\right),$$

subject to the initial conditions $x\left(0\right) = 1$, $y\left(0\right) = 0$, $\dot{x}\left(0\right) = 0$.

2. (a) Calculate the inverse Laplace transform below using, the method of partial fractions, and then the Convolution Theorem:

$$\mathrm{L}^{-1}\left[\frac{2}{s^2\left(s^2+4\right)}\right].$$

Verify your answers by Laplace transforming the resulting function.

(b) Solve the following initial value problem (IVP):

$$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + \left(a^2+b^2\right)y\left(t\right) = 0,$$

subject to the conditions $y\left(0\right) = 0$, $\dot{y}\left(0\right) = 1$, assuming $a, b > 0$.

3. The function $f\left(x\right)$ is defined, on the interval $-2 \leqslant x < 2$, by

$$f\left(x\right) = x^2 + 1,$$

and elsewhere by periodicity, $f\left(x + 2\pi\right) = f\left(x\right)$.

(a) Find the Fourier Series of $f\left(x\right)$.

(b) Sketch the function $f\left(x\right)$ on the interval $-6 \leqslant x < 6$.

(c) State the value to which the series converges at $x = 0$. Hence prove that:

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

4. (a) For the functions $x\left(x^2+y^2\right)^{-1}$, $v\left(x,y\right)=-y\left(x^2+y^2\right)^{-1}$, show that

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{AND} \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

Hence or otherwise, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{AND} \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0.$$

(b) If the cartesian coordinates x and y are related to the plane polar coordinates r and θ by $x=r\cos\left(\theta\right)$ and $y=r\sin\left(\theta\right)$, prove that

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}.$$

5. (a) Show that the following function

$$f\left(x,y\right)=x^2+y^2+\frac{1}{x^2y^2}$$

has four stationary points. Investigate the nature of these points.

(b) Use the method of Lagrange Multipliers to find the minimum distance between the origin and a point on the plane

$$x+2y+2z=3.$$

TABLE OF LAPLACE TRANSFORMS

In all cases herein, a is a constant, and n is a positive integer:

$f(t) = \mathrm{L}^{-1}\left[\overline{f(s)}\right]$	$\overline{f(s)} = \mathrm{L}[f(t)]$
1	$\dfrac{1}{s}$
e^{at}	$\dfrac{1}{s-a}$
t^n	$\dfrac{n!}{s^{n+1}}$
$\sin(at)$	$\dfrac{a}{s^2+a^2}$
$\cos(at)$	$\dfrac{s}{s^2+a^2}$
$\mathrm{H}(t-a)$	$\dfrac{\exp[-as]}{s}$
$\delta(t-a)$	$\exp[-as]$

The Heaviside function, $\mathrm{H}(t-a)$, is defined by

$$\boxed{\mathrm{H}(t-a) = \begin{cases} 0 & \text{FOR } 0 \leqslant t < a, \\ 1 & \text{FOR } t \geqslant a. \end{cases}}$$

(In this context, the constant a is understood to be positive).