Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway

GX 2143

Semester I Examinations 2003/2004

Exam Code(s)	2BA1; 2CS2; 2BE1; 2BN1; 2BP1; 3BS1; 3CS1; 3CS2; 3EL1; 3BS9.
Exam(s)	Second Science, Arts and Engineering
Module Code(s)	MM245
Module(s)	Numerical Analysis
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor B. Straughan
Internal Examiner(s)	Dr. M. Ó Confhaola Dr. P.M. O'Leary

Instructions: Full marks for THREE completed questions.
Relevant NUMERICAL FORMULAE are listed at the end of this paper

Duration	2hrs
No. of Answer books	1

Requirements

Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	3
Department(s)	Mathematical Physics

1. Show that the equation $f(x) = x^3 - 7x + 5 = 0$ has a solution in theinterval [0, 1]. Consider the following two iterative schemes $x_{n+1} = g(x_n)$ for finding a root to this equation:

   (i) $g(x) = (x^3 + 5)/7$

   (ii) $g(x) = (7x - 5)^{1/3}$.

   (a) Choosing $x_0 = 0.8$ and retaining three places of decimals, calculate the root $x = s$ of $f(x) = 0$ in [0, 1] by trying schemes (i) and (ii) above.

   (b) Calculate the values of $g'(s)$ for each of the two schemes and comment on the results of (a) in the light of these values.

   (c) Choosing $x_0 = 0.8$ and retaining four places of decimals,calculate the root $x = s$ using the Newton-Raphson scheme.

2. The equation $f(x) = 8x^3 + 12x^2 - 18x - 11 = 0$ has a solution $s = -1/2$. Show that the following is a valid scheme for finding a root of this equation:

$$x_{n+1} = x_n^3/3 + x_n^2/2 + x_n/4 - 11/24$$

   (a) Show that this scheme is third order for the root $s = -1/2$.

   (b) Choosing $x_0 = -1$ for the above scheme and retaining six places of decimals, calculate the numerical convergence towards $s = -1/2$.

   (c) Set up the Newton-Raphson scheme for finding a root of $f(x) = 0$. Choosing $x_0 = -1$ and retaining six places of decimals, calculate the numerical convergence towards $s = -1/2$.

3. Use cubic Lagrangian interpolation to estimate $f(4.5)$using the following function values.

x	3.1	4.0	5.1	6.0
$f(x)$	1.131402	1.386294	1.629241	1.791759

Given that $f(x) = \ln(x)$, find the upper and lower bounds for the error at $x = 4.5$. Confirm that the actual error in your estimate for $f(4.5)$ lies between these bounds.

4. Construct a forward difference table from the following data for a function $f(x)$.

x	0.5	0.6	0.7	0.8
$f(x)$	0.479426	0.564642	0.644218	0.717356

We denote by $p_1(x)$ the polynomial passing through the first two data points, by $p_2(x)$ the polynomial passing through the first three data points, and by $p_3(x)$ the polynomial passing through all four data points.
Construct $p_1(x)$, $p_2(x)$ and $p_3(x)$ using Newton's forward difference formula and use them to obtain three estimates for $f(0.73)$. Given that $f(x) = \sin(x)$, find the upper and lower bounds for the errors at $x = 0.73$. Confirm that the actual error lies between these bounds.

BASIC EQUATIONS

1. The Newton-Raphson scheme for solving the equation $f(x) = 0$ is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ choose an } x_0$$

The secant method for solving the equation $f(x) = 0$ is given by

$$x_{n+1} = x_n - f(x_n)\{\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\}, \text{choose } x_0, x_1.$$

2. A function $f(x)$ defined on $[a,b]$ passes through the $n+1$ points (x_0,f_0), (x_1,f_1), ,,,,,,(x_n,f_n) where $a = x_0, b = x_n$ and $x_i < x_j$ for $i < j$. The Lagrangian form for the polynomials $p_n(x)$ passing through these $n+1$ points is given by

$$p_n(x) = \sum_{k=0}^{n} L_k(x) f_k$$

where

$$L_k(x) = \frac{(x-x_0)(x-x_1)(x-x_2)..(x-x_{k-1})(x-x_{k+1})..(x-x_n)}{(x_k-x_0)(x_k-x_1)(x_k-x_2)..(x_k-x_{k-1})(x_k-x_{k+1})..(x_k-x_n)}$$

If the points are equally spaced with $x_{i+1} - x_i = h > 0, h$ constant, then Newton's forward difference form for $p_n(x)$ is, in the usual notation,

$$p_n(x) = f_0 + \binom{r}{1}\Delta f_0 + \binom{r}{2}\Delta^2 f_0 + \ldots.. + \binom{r}{n}\Delta^n f_0$$

where $r = (x - x_0)/h$.
The error in the interpolation $e_n(x) = f(x) - p_n(x)$ is given by

$$e_n(x) = (x-x_0)(x-x_1)(x-x_2)....(x-x_n)\frac{f^{n+1}(t)}{(n+1)!}$$

where t lies in $[x_0, x_n]$.