

Ollscoil na hÉireann, Gaillimh
National University of Ireland, Galway
Christmas Examinations, 2003/2004

GX 2148

Exam Code(s) 2FM1.
Exam(s) FINANCIAL MATHEMATICS AND ECONOMICS 2ND YEAR
Module Code(s) MP291
Module(s) **Mathematical Methods II**

Paper No.
Repeat Paper
Special Paper

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Instructions: Attempt not more than *THREE* questions.
Duration *TWO* hours
No. of Answer books _____

Requirements: _____
Handout _____
MCQ 4
Statistical Tables YES, LOG TABLES
Graph Paper _____
Log Graph Paper _____
Other Material _____
No. of Pages 3 PAGES (Excluding Cover Page)
Department(s) MATHEMATICAL PHYSICS

1. (a) Prove the '*Differential*' theorem for Laplace transforms

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} (\mathcal{L}[f(t)]) .$$

- (b) Prove the *Second Shift theorem* for Laplace transforms

$$\mathcal{L}[H(t-a) f(t-a)] = e^{-as} \mathcal{L}[f(t)] ,$$

where a is an arbitrary positive constant.

- (c) Use Laplace transforms to solve the following set of simultaneous differential equations

$$\begin{aligned} \frac{dx}{dt} + 4y(t) &= \sin(3t) , \\ \frac{d^2x}{dt^2} + 2\frac{dy}{dt} + 4x(t) &= \cos(3t) , \end{aligned}$$

subject to the initial conditions $x(0) = 1, y(0) = 0, \dot{x}(0) = 0$.

2. (a) Calculate the inverse Laplace transform below using, the method of partial fractions, and then the Convolution Theorem:

$$\mathcal{L}^{-1} \left[\frac{2}{s^2(s^2 + 4)} \right] .$$

Verify your answers by Laplace transforming the resulting function.

- (b) Solve the following initial value problem (IVP):

$$\frac{d^2y}{dt^2} - 2a\frac{dy}{dt} + (a^2 + b^2)y(t) = 0 ,$$

subject to the conditions $y(0) = 0, \dot{y}(0) = b$, assuming $a, b > 0$.

3. The function $f(x)$ is defined, on the interval $-2 \leq x < 2$, by

$$f(x) = x^2 + 1 ,$$

and elsewhere by periodicity, $f(x + 2\pi) = f(x)$.

- (a) Find the Fourier Series of $f(x)$.
 (b) Sketch the function $f(x)$ on the interval $-6 \leq x < 6$.
 (c) State the value to which the series converges at $x = 0$. Hence prove that:

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots$$

4. (a) Show that the following function

$$f(x, y) = x^2 + y^2 + \frac{1}{x^2 y^2}$$

has four stationary points. Investigate the nature of these points.

- (b) Use the method of Lagrange Multipliers to find the minimum distance between the origin and a point on the plane

$$x + 2y + 2z = 3.$$

5. Find all equilibrium solutions of the system of equations

$$\begin{aligned}\frac{dx}{dt} &= 1 - xy, \\ \frac{dy}{dt} &= x - y^3.\end{aligned}$$

and determine (if possible) whether they are stable or unstable.

OR

Consider the boundary value problem, where $u(x, t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad \text{FOR } 0 < x < 1, \quad t > 0$$

subject to

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(1, t) = 0, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \quad \text{AND} \quad u(x, 0) = x.$$

Find the solution using separation of variables.

TABLE OF LAPLACE TRANSFORMS

In all cases herein, a is a constant, and n is a positive integer:

$f(t) = L^{-1} [\overline{f(s)}]$	$\overline{f(s)} = L[f(t)]$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2 + a^2}$
$\cos(at)$	$\frac{s}{s^2 + a^2}$
$H(t-a)$	$\frac{\exp[-as]}{s}$
$\delta(t-a)$	$\exp[-as]$

The Heaviside function, $H(t-a)$, is defined by

$$H(t-a) = \begin{cases} 0 & \text{FOR } 0 \leq t < a, \\ 1 & \text{FOR } t \geq a. \end{cases}$$

(In this context, the constant a is understood to be positive).