Ollscoil na hÉireann, Galway GX 2150
National University of Ireland, Galway

Semester I Examinations, 2003/2004

Exam Code(s)	4CS2, 3PT2, 3PT1, 3EL1
Exam(s)	Fourth Science, Third Science
Module Code(s)	MP332
Module(s)	Calculus of Variations
Paper No	1
Repeat Paper	
Special Paper	
External Examiner(s)	Professor Brian Straughan
Internal Examiner(s)	Dr. Micheál Ó Confhaola
	Dr. M.G. Meere
Instructions:	Attempt **THREE** questions
Duration	**TWO HOURS**
No. of Answer books	
Requirements	
Handout	
MCQ	
Statistical Tables	Yes - Log Tables
Graph paper	
Log Graph Paper	
Other Material	
No. of Pages	2 (excluding this page)
Department(s)	Mathematical Physics

1. a. Determine the $C^2[0,1]$ curve $y = y(x)$ with fixed end points $(0,0)$, $(1,1)$ which gives

$$J = \int_0^1 (y')^2/x^3 dx$$

a minimum value, assuming a minimum exists. What is the minimum value?

b. Find the extremal $y = y(x), y \in C^4[0,1]$ for the following functional satisfying the given boundary conditions:

$$J(y) = \int_0^1 (yy' + (y'')^2)dx \text{ with } y(0) = 0,\ y'(0) = 1,\ y(1) = 2,\ y'(1) = 4.$$

2. a. Consider the problem of finding the extremal $y = y(x), y \in C^2[a,b]$ of the functional

$$J(y) = \int_a^b F(x,y,y')dx$$

subject to

$$y(a) = y_0, y(b) \text{ unspecified},$$

with a, b, y_0 given constants. You may assume that F is twice continuously differentiable in each of its three arguments. Show that the boundary condition on $x = b$ for the extremal is given by

$$\frac{\partial F}{\partial y'} = 0 \text{ on } x = b.$$

b. Find the extremal $y = y(x), y \in C^2[0,1]$ for

$$J(y) = \int_0^1 ((y')^2 + y^2)dx$$

subject to

$$y(0) = 1, y(1) \text{ unspecified}.$$

3. a. Let $y = y(x), y \in C^2[0,l]$ be the unique solution of the two point boundary value problem

$$\frac{d}{dx}\left(T(x)\frac{dy}{dx}\right) - k(x)y = -w(x),\ 0 < x < l,$$
$$y(0) = y(l) = 0,$$

where $T(x) \in C^1[0,l]$ and $T(x) > 0$ in $[0,l]$, and where $k(x), w(x) \in C[0,l]$ and $k(x) > 0,\ w(x) \geq 0$ in $[0,l]$. Show that

$$\int_0^l \left\{\frac{(U' + w)^2}{k} + \frac{U^2}{T}\right\}dx \geq \int_0^l wydx,$$

where U is any $C^1[0,l]$ function.

b. Consider the boundary value problem:

$$y'' - y = -1,\ 0 < x < \pi/2,$$
$$y(0) = y'(\pi/2) = 0.$$

Noting the bounds

$$\int_0^{\pi/2} \{(U' + 1)^2 + U^2\}dx \geq \int_0^{\pi/2} ydx \geq \int_0^{\pi/2} \{2Y - Y'^2 - Y^2\}dx$$

where $Y(0) = 0$, $U(\pi/2) = 0$ and U, Y are sufficiently smooth, calculate the 'best' Y of the type $Y = \beta \sin x$. Calculate the 'best' U of the type $U = \gamma \cos x$. Compute the corresponding bounds for $\int_0^{\pi/2} ydx$.

4. **a.** Find extremals $y = y(x)$, $y \in C^2[0,\pi]$ for the isoperimetric problem

$$J(y) = \int_0^{\pi} (y')^2 dx,\ y(0) = y(\pi) = 0,$$

subject to

$$\int_0^{\pi} y^2 dx = 1.$$

b. Find the curve $y = y(x), y \in C^2(0,1)$, with fixed arclength $l \geq \pi/2$, passing through the points $(0,0)$ and $(1,0)$ which has maximal area between it and the x-axis, $0 \leq x \leq 1$.