GX 2153

# *Ollscoil na hÉireann, Gaillimh*
# *National University of Ireland, Galway*

## Semester One Examinations, 2003/2004

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| Exam Code(s) | 4PT2 |
| Exam(s) | Fourth Science |
| Module Code(s) | MP403 |
| Module(s) | Cosmology and General Relativity |
| Paper No | 1 |
| Repeat Paper | |
| Special Paper | |
| External Examiner(s) | Professor B. Straughan |
| Internal Examiner(s) | Dr. M. Ó Confhaola |
| | Dr. T.N. Sherry |

**Instructions:** Answer: THREE questions.
All questions carry the same marks.

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| Duration | 2 hours |
| No. of Answer books | 1 |
| **Requirements** | |
| Handout | |
| MCQ | |
| Statistical Tables | |
| Graph paper | |
| Log Graph Paper | |
| Other Material | |
| No. of Pages | 3 |
| Department(s) | Mathematical Physics |

1. a. The metric for an isotropic 3-dimensional space is given by the distance between neighbouring points

$$ds^2 = \frac{dr^2}{1-4r^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

In this space determine [ see general formulae at end of paper ]

i. the Gauss curvature $K(r)$ ;

ii. the proper radius of the $r$-sphere $r = R$ ;

iii. the surface area of the $r$-sphere $r = R$ , and

iv. the volume enclosed by the $r$-sphere $r = R$ .

b. Repeat the four computations of part (a) for the isotropic space with metric

$$ds^2 = (1 - kr^2)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

In this case you are to discuss the properties of the curvature $K(r)$ for the cases $k < 0$, $k = 0$ and $k > 0$.

2. a. Explain *briefly* what a timelike geodesic is in spacetime and the role such geodesics play in General Relativity.

b. The Schwarzschild spacetime outside a spherical star of mass $M$ is endowed with the metric

$$d\tau^2 = \left(1 - 2\frac{GM}{c^2 r}\right)dt^2 - \frac{1}{c^2}\left[\left(1 - 2\frac{GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right].$$

Derive the equations for a timelike geodesic in the equatorial "plane" $\theta = \frac{\pi}{2}$ and show that they lead to an orbit in the equatorial plane described by the equation

$$\frac{d^2 u}{d\phi^2} + u - 3\,\varepsilon\, u^2 = \frac{GM}{B^2}$$

where $u(\phi) = 1/r(\phi)$, $\varepsilon = GM/c^2$ is very small and $B$ is a constant.

c. Show that a first order perturbative solution of this equation is given by

$$u(\phi) = \frac{GM}{B^2} + a\cos\phi + \varepsilon\left[2a^2 + 3\frac{G^2M^2}{B^4} - a^2\cos^2\phi + 3a\frac{GM}{B^2}\,\phi\sin\phi\right]$$

and comment on the implications of the final term in this solution.

3. a. Modern cosmology assumes a uniform universe. Explain how the assumptions of homogeneity and isotropy lead to the Robertson-Walker metric

$$d\tau^2 = dt^2 - \frac{R(t)^2}{c^2}\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2\, d\theta^2 + \sigma^2 \sin^2\theta \, d\phi^2\right]$$

in terms of comoving coordinates $\{t, \sigma, \theta, \phi\}$. Show how Hubble's law arises in this approach.

b. Show, using the Robertson-Walker metric in part (a), that in an expanding universe light received from distant galaxies is red-shifted, while in a contracting universe the light is blue-shifted.

c. Distinguish between the concepts of an object horizon $\sigma_{oh}$ and an event horizon $\sigma_{eh}$ and explain how they may occur in uniform universe models of cosmology.

4. Show that the Gauss curvature of a 2-dimensional ( $\theta =$ constant, $\phi =$ constant ) Robertson-Walker spacetime is given by

$$K(t) = -\frac{d^2R}{dt^2}/R(t)$$

and use this to derive the Friedmann equation which expresses the relationship between the cosmic scale factor $R(t)$ and matter namely

$$\frac{d^2R}{dt^2} = -\frac{4\pi}{3}\frac{G\rho(t_0)R^3(t_0)}{R^2(t)} + \frac{\Lambda}{3}R(t)\,.$$

Comment on the role of the cosmological constant $\Lambda$ in this equation.
The integrated form of the Friedmann equation is found to be

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi}{3}\frac{G\rho(t_0)R^3(t_0)}{R(t)} - kc^2 + \frac{\Lambda}{3}R^2(t)\,.$$

Hence show qualitatively that

a. models with $\Lambda < 0$ go through an expansion from an initial infinite density big bang followed by a contraction down to a final big implosion;

b. models with $\Lambda = 0$ also expand from an initial big bang, but may be able to avoid the final implosion;

c. models with $0 < \Lambda < \Lambda_c$ for some critical value of $\Lambda$ need not necessarily suffer a big bang in their past, provided $k = 1$; such models may go through a contraction phase from an initial non-zero value $R(0)$ followed by an expansionary phase without ever hitting the singular behaviour;

d. models with $\Lambda > \Lambda_c$ expand from an initial big bang and continue to expand.

## Useful Formulae

1. In a 2-dimensional space with orthogonal coordinates the Gauss curvature is

$$K(x^1,x^2) = \frac{1}{2\,g_{11}g_{22}}\left\{-\frac{\partial^2 g_{11}}{\partial(x^2)^2} - \frac{\partial^2 g_{22}}{\partial(x^1)^2} + \frac{1}{2\,g_{11}}\left[\frac{\partial g_{11}}{\partial x^1}\frac{\partial g_{22}}{\partial x^1} + \left(\frac{\partial g_{11}}{\partial x^2}\right)^2\right] + \frac{1}{2\,g_{22}}\left[\frac{\partial g_{11}}{\partial x^2}\frac{\partial g_{22}}{\partial x^2} + \left(\frac{\partial g_{22}}{\partial x^1}\right)^2\right]\right\}\,.$$

The element of surface area in the $x^3$ = constant surface of a 3-dimensional space with orthogonal coordinates is

$$dA = \sqrt{g_{11}\,g_{22}}\;dx^1dx^2$$

while the volume element is

$$dV = \sqrt{g_{11}\,g_{22}\,g_{33}}\;dx^1dx^2dx^3\;.$$